“Two are better than one: volatility forecasting using multiplicative component GARCH-MIDAS models”

Online Appendix

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Contents

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A Proofs

Proof of Proposition 1. The proof follows directly by applying the mutual independence of $g_{i,t}$, τ_t and $Z_{i,t}$ and by noting that Assumption 3 implies $\mathbf{E}[\tau_t^2]/\mathbf{E}[\tau_t]^2 > 1$ if τ_t is non-constant. ∎

Proof of Proposition 2. First, note that under Assumptions 1, 2, and 3 the covariance $\mathbf{Cov}(\varepsilon_t^2, \varepsilon_{t-k}^2)$ exists for every $k \in \mathbb{N}$ and is time-invariant. In the proof, we use that τ_t and g_t are independent covariance stationary processes and that Z_t are *i.i.d.* innovations.

$$
\begin{aligned}
\rho_k^{MG}(\varepsilon^2) &= \frac{\mathbf{Cov}(\varepsilon_t^2, \varepsilon_{t-k}^2)}{\sqrt{\mathbf{Var}(\varepsilon_t^2)}\sqrt{\mathbf{Var}(\varepsilon_{t-k}^2)}} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}]\mathbf{E}[g_t Z_t^2 g_{t-k} Z_{t-k}^2] - \mathbf{E}[\tau_t]\mathbf{E}[\tau_{t-k}]}{\mathbf{Var}(\varepsilon_t^2)} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}]\mathbf{E}[g_t Z_t^2 g_{t-k} Z_{t-k}^2] - \mathbf{E}[\tau_t\tau_{t-k}] + \mathbf{E}[\tau_t\tau_{t-k}] - \mathbf{E}[\tau_t]\mathbf{E}[\tau_{t-k}]}{\mathbf{Var}(\varepsilon_t^2)} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}] - \mathbf{E}[\tau_t]^2}{\mathbf{Var}(\varepsilon_t^2)} + \frac{\left(\mathbf{E}[g_t Z_t^2 g_{t-k} Z_{t-k}^2] - \mathbf{E}[g_t]\mathbf{E}[g_{t-k}]\right)\mathbf{E}[\tau_t\tau_{t-k}]}{\mathbf{Var}(\varepsilon_t^2)} \\
&= \frac{\mathbf{Cov}(\tau_t, \tau_{t-k})}{\mathbf{Var}(\varepsilon_t^2)} + \frac{\mathbf{Cov}(g_t Z_t^2, g_{t-k} Z_{t-k}^2)(\mathbf{Cov}(\tau_t, \tau_{t-k}) + \mathbf{E}[\tau_t^2])}{\mathbf{Var}(\varepsilon_t^2)} \\
&= \rho_k^{\tau}\frac{\mathbf{Var}(\tau_t)}{\mathbf{Var}(\varepsilon_t^2)} + \rho_k^{GA}\frac{\left(\rho_k^{\tau}\mathbf{Var}(\tau_t) + \mathbf{E}[\tau_t]^2\right)\mathbf{Var}(g_t Z_t^2)}{\mathbf{Var}(\varepsilon_t^2)}
\end{aligned}
$$

∎

Proof of Proposition 3. Employing the assumptions used in the proof of Proposition 2 above, we conclude similarly:

$$
\begin{aligned}
\rho_k^{MG}(\sigma^2) &= \frac{\mathbf{Cov}(\sigma_t^2, \sigma_{t-k}^2)}{\sqrt{\mathbf{Var}(\sigma_t^2)}\sqrt{\mathbf{Var}(\sigma_{t-k}^2)}} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}]\mathbf{E}[g_t g_{t-k}] - \mathbf{E}[\tau_t]\mathbf{E}[\tau_{t-k}]}{\mathbf{Var}(\sigma_t^2)} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}]\mathbf{E}[g_t g_{t-k}] - \mathbf{E}[\tau_t\tau_{t-k}] + \mathbf{E}[\tau_t\tau_{t-k}] - \mathbf{E}[\tau_t]\mathbf{E}[\tau_{t-k}]}{\mathbf{Var}(\sigma_t^2)} \\
&= \frac{\mathbf{E}[\tau_t\tau_{t-k}] - \mathbf{E}[\tau_t]^2}{\mathbf{Var}(\sigma_t^2)} + \frac{(\mathbf{E}[g_t g_{t-k}] - \mathbf{E}[g_t]\mathbf{E}[g_{t-k}])\,\mathbf{E}[\tau_t\tau_{t-k}]}{\mathbf{Var}(\sigma_t^2)} \\
&= \frac{\mathbf{Cov}(\tau_t, \tau_{t-k})}{\mathbf{Var}(\sigma_t^2)} + \frac{\mathbf{Cov}(g_t, g_{t-k})(\mathbf{Cov}(\tau_t, \tau_{t-k}) + \mathbf{E}[\tau_t^2])}{\mathbf{Var}(\sigma_t^2)} \\
&= \rho_k^{\tau}\frac{\mathbf{Var}(\tau_t)}{\mathbf{Var}(\sigma_t^2)} + \rho_k^{g}\frac{\left(\rho_k^{\tau}\mathbf{Var}(\tau_t) + \mathbf{E}[\tau_t]^2\right)\mathbf{Var}(g_t)}{\mathbf{Var}(\sigma_t^2)}
\end{aligned}
$$

∎

Proof of Proposition 4. Eq. (16) follows directly from the mutual independence of $g_{i,t}$, τ_t, and $Z_{i,t}$. Next, eq. (17) is derived as

$$
\begin{aligned}
\mathbf{E}[g^2_{k,t+1|t}] &= \mathbf{E}\left[\left(1+(\alpha+\gamma/2+\beta)^{k-1}(g_{1,t+1|t}-1)\right)^2\right] \\
&= 1+2(\alpha+\gamma/2+\beta)^{k-1}\underbrace{(\mathbf{E}[g_{1,t+1|t}]-1)}_{=0}+(\alpha+\gamma/2+\beta)^{2(k-1)}(\mathbf{E}[g^2_{1,t+1|t}]-1) \\
&= 1+(\alpha+\gamma/2+\beta)^{2(k-1)}(\mathbf{E}[g^2_{1,t+1}]-1).
\end{aligned}
$$

In the last step, we use that $g_{1,t+1|t} = g_{1,t+1}$. Now, consider the first property: As $k \to \infty$, $\mathbf{E}[g^2_{k,t+1|t}]$ decreases monotonically towards one. Because the numerator decreases while the denominator is constant, R^2_k is decreasing in k. The limit follows readily from $\lim_{k\to\infty} \mathbf{E}[g^2_{k,t+1|t}] = 1$.

For deriving the second property, note that eq. (16) is a rational function of linear polynomials in $\mathbf{E}[\tau^2_{t+1}]$ with negative intercepts and positive gradients. By taking the first derivative, the signs of intercepts and gradients imply the rational function in $\mathbf{E}[\tau^2_{t+1}]$ to be strictly increasing.

∎

Proof of Lemma 1. Using eq. (4), we obtain

$$
\begin{aligned}
R^2_1 &= \frac{\mathbf{Var}(g_t\tau_t)}{\mathbf{Var}(\varepsilon^2_t)} = \frac{\mathbf{E}[g^2_t]\mathbf{E}[\tau^2_t]-\mathbf{E}[\tau_t]^2}{\mathbf{E}[g^2_t]\mathbf{E}[\tau^2_t]\kappa-\mathbf{E}[\tau_t]^2} \\
&= \frac{(1-(\alpha+\gamma/2+\beta)^2)\mathbf{E}[\tau^2_t]-(1-(\alpha+\gamma/2)^2\kappa-2(\alpha+\gamma/2)\beta-\beta^2)\mathbf{E}[\tau_t]^2}{(1-(\alpha+\gamma/2+\beta)^2)\mathbf{E}[\tau^2_t]\kappa-(1-(\alpha+\gamma/2)^2\kappa-2(\alpha+\gamma/2)\beta-\beta^2)\mathbf{E}[\tau_t]^2}.
\end{aligned}
$$

∎

B Additional Tables

Table B.1: Monte-Carlo parameter estimates of MS-GARCH-TVI.

	ω_1	ω_2	α	β	$p_{1,1}$	$p_{2,2}$
Panel A: $Z_{n,i,t}$ normally distributed						
Monthly τ_t	0.029	0.050	0.057	0.910	0.997	0.995
	[0.024,0.034]	[0.038,0.067]	[0.053,0.062]	[0.902,0.917]	[0.992,0.999]	[0.982,0.998]
Daily τ_t	0.020	0.038	0.058	0.912	0.993	0.991
	[0.016,0.024]	[0.029,0.051]	[0.054,0.063]	[0.906,0.919]	[0.986,0.997]	[0.978,0.996]
Panel B: $Z_{n,i,t}$ student-t distributed						
Monthly τ_t	0.028	0.066	0.052	0.914	0.993	0.980
	[0.021,0.035]	[0.050,0.088]	[0.045,0.058]	[0.904,0.925]	[0.984,0.997]	[0.941,0.994]
Daily τ_t	0.019	0.050	0.053	0.917	0.990	0.978
	[0.015,0.024]	[0.038,0.066]	[0.046,0.059]	[0.907,0.925]	[0.980,0.995]	[0.946,0.990]

Notes: The table reports the median MS-GARCH-TVI parameter estimates and in brackets the corresponding inter-quartile ranges across 2,000 Monte-Carlo simulations in which the true data-generating process is a GARCH-MIDAS model, see description of Table 1.

Table B.2: Summary statistics of stock market returns and explanatory variables.

Variable	Freq.	Start	Obs.	Min.	Max.	Mean	Median	Sd.	Skew.	Kurt.
Stock market data										
S&P 500 returns	d	1971	11938	-22.90	10.96	0.03	0.04	1.06	-1.04	28.81
$\sqrt{\text{RV}}$	d	2000	4600	0.13	8.84	0.87	0.72	0.60	3.22	21.93
RVol(22)	d	1989	7390	0.23	5.54	0.95	0.80	0.56	2.97	17.46
Explanatory variables										
VIX	d	1990	7135	0.58	5.09	1.22	1.10	0.49	2.08	10.63
NFCI	w	1973	2470	-0.99	4.67	0.00	-0.33	1.00	1.94	6.53
NAI	m	1971	568	-5.16	2.76	-0.00	0.06	1.00	-1.21	6.96
Δ IP	m	1971	568	-4.43	2.38	0.18	0.22	0.72	-1.22	8.82
Δ Housing	m	1971	568	-30.67	25.67	-0.07	-0.19	8.03	-0.03	3.77

Notes: The table presents summary statistics for the different variables, whereby the column "Freq." indicates whether the data is observed on a daily (d), weekly (w) or monthly (m) frequency. The column "Start" indicates the year of the first observation for each variable. The data end in 2018:M4. The reported statistics include the number of observations ("Obs."), the minimum ("Min.") and maximum ("Max."), the mean and median, the standard deviation ("Sd."), the skewness ("Skew.") and the kurtosis ("Kurt."). We define $RVol(22)_{i,t} = \sqrt{1/22 \sum_{j=0}^{21} r_{i-j,t}^2}$. Changes in industrial production and housing starts are measured in month-over-month log differences, i.e. $\Delta X_t = 100 \cdot (\ln(X_t) - \ln(X_{t-1}))$.

Table B.3: Full sample estimates of competitor models.

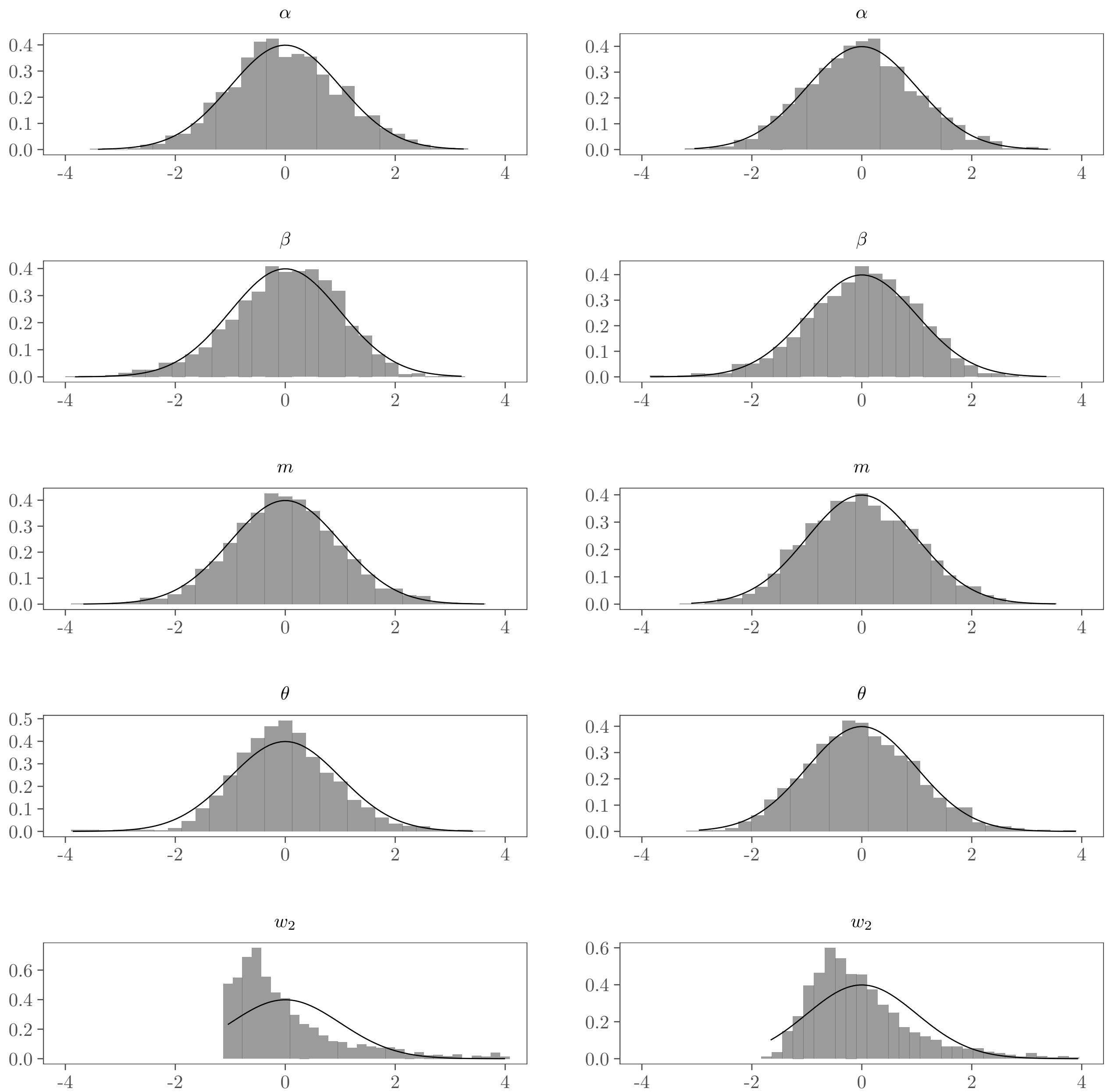
Latent variables	Observables	Distribution
MS-GARCH-TVI (Haas et al., 2004)		
Regime 1: $\tilde{\sigma}^2_{1,t} = 0.008 + (0.018 + 0.092\mathbb{1}_{\{\varepsilon_{t-1}<0\}})\varepsilon^2_{t-1} + 0.918\tilde{\sigma}^2_{1,t-1}$ *Regime 2:* $\tilde{\sigma}^2_{2,t} = 0.183 + (0.018 + 0.092\mathbb{1}_{\{\varepsilon_{t-1}<0\}})\varepsilon^2_{t-1} + 0.918\tilde{\sigma}^2_{1,t-1}$ *Markov chain:* $X_t \in \{1,2\}$ with trans. prob. $p_{1,1} = 0.955$ and $p_{2,2} = 0.153$	$\varepsilon_t = \tilde{\sigma}_{X_t,t} Z_t$	$Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$
MS-GARCH-TVC (Haas et al., 2004)		
Regime 1: $\tilde{\sigma}^2_{1,t} = 0.006 + (0.016 + 0.077\mathbb{1}_{\{\varepsilon_{t-1}<0\}})\varepsilon^2_{t-1} + 0.930\tilde{\sigma}^2_{1,t-1}$ *Regime 2:* $\tilde{\sigma}^2_{2,t} = 1.067 + 0.559\varepsilon^2_{t-1} + 0.439\tilde{\sigma}_{2,t-1}$ *Markov chain:* $X_t \in \{1,2\}$ with trans. prob. $p_{1,1} = 0.946$ and $p_{2,2} = 0.096$	$\varepsilon_t = \tilde{\sigma}_{X_t,t} Z_t$	$Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$
Realized GARCH (Hansen et al., 2012)		
$\log \sigma^{RG}_t = 0.170 + 0.373 \log RV^{int}_{t-1} + 0.575 \log \sigma^{RG}_{t-1}$	$r_t - 0.018 = \sqrt{\sigma^{RG}_t} Z_t$ $\log RV^{int}_t = -0.472 + 1.056 \log \sigma^{RG}_t - 0.102 Z_t + 0.117\left((Z_t)^2 - 1\right) + u_t$	$Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ $u_t \overset{i.i.d.}{\sim} \mathcal{N}(0,0.537)$
HEAVY (Shephard and Sheppard, 2010)		
$\sigma^{HVY}_t = 0.025 + 0.496 \cdot RV^{int}_{t-1} + 0.611 \cdot \sigma^{HVY}_{t-1}$ $\sigma^{RV^{int}}_t = 0.017 + 0.464 \cdot RV^{int}_{t-1} + 0.536 \cdot \sigma^{RV^{int}}_{t-1}$	$\varepsilon_t = \sqrt{\sigma^{HVY}_t} Z_t$ $RV^{int}_t = \sigma^{RV^{int}}_t Z^2_{RV^{int},t}$	$Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ $Z_{RV^{int},t} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$
HAR (Corsi, 2009)		
	$\log(RV_{t+1}) = -0.117 + 0.353 \log RV_t + 0.380 \log\left(\frac{RV_{t-4:t}}{5}\right) + 0.211 \log\left(\frac{RV_{t-21:t}}{22}\right) + \zeta_t$	$\mathbf{E}[\zeta_{t,k} \mid \mathcal{F}_{t-1,k}] = 0$
HAR (lev.) (Corsi and Reno, 2012)		
	$\log(RV_{t+1}) = -0.142 + 0.260 \log RV_t + 0.365 \log\left(\frac{RV_{t-4:t}}{5}\right) + 0.290 \log\left(\frac{RV_{t-21:t}}{22}\right)$ $-0.078 r_t - 0.208 \frac{r_{t-4:t}}{5} - 0.155 \frac{r_{t-21:t}}{22} + \zeta^{lev}_t$	$\mathbf{E}[\zeta^{lev}_{t,k} \mid \mathcal{F}_{t-1,k}] = 0$

Notes: Benchmark models introduced in Section 4.4.1 and their corresponding full sample estimates. r_t, RV^{int}_t, and RV_t denote the return, the intraday realized variance and its overnight return-augmented close-close equivalent. All benchmark models assume $I_t = 1$, i.e. they are defined on a daily frequency. Distributional assumptions are those employed for estimation, e.g. normal innovations for QML estimation.

C Additional Figures

Figure C.1: Histograms of standardized GARCH-MIDAS parameter estimates.

(a) Monthly τ_t (b) Daily τ_t



Notes: Standardized empirical distributions of parameter estimates across 2000 simulations are reported. On the left, the underlying data is generated by a GARCH-MIDAS model with monthly varying τ_t, on the right with daily varying τ_t, see Section 3 for further details. The standard normal distribution is depicted in black.

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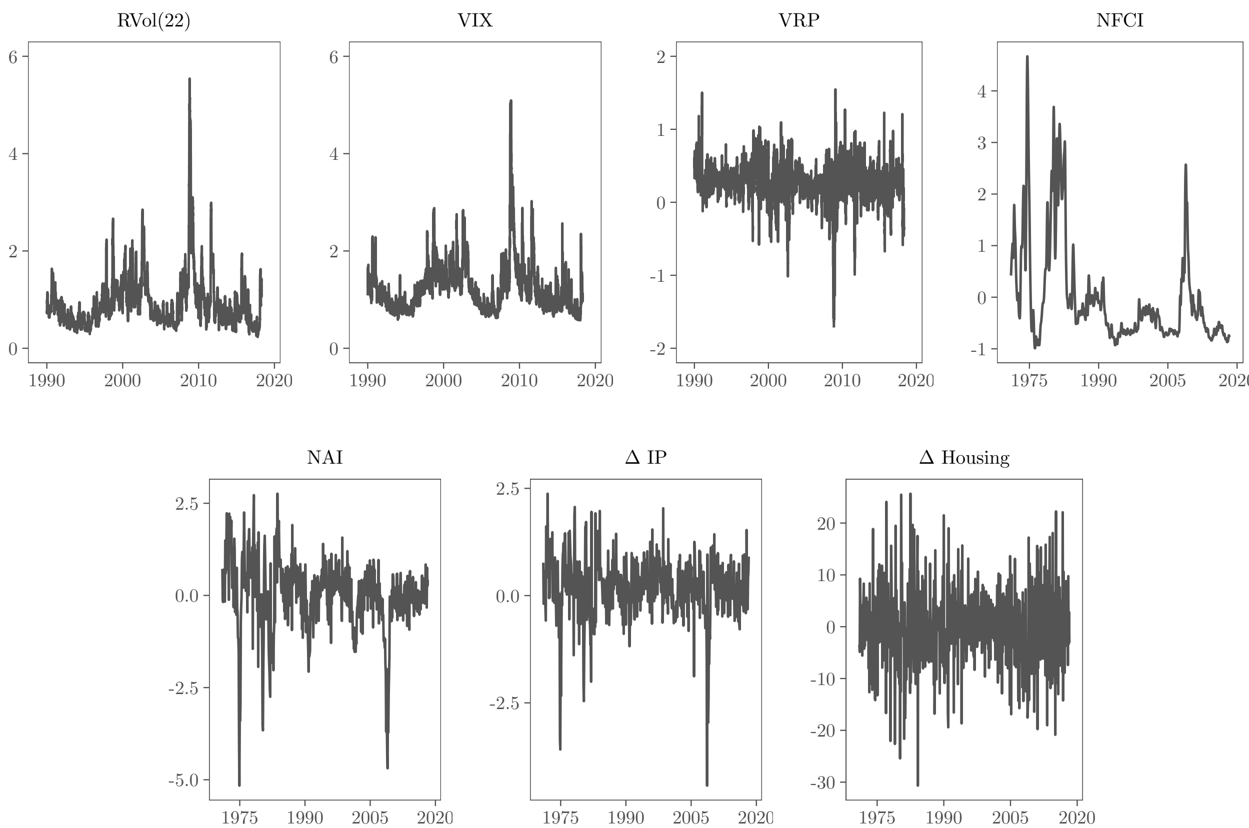
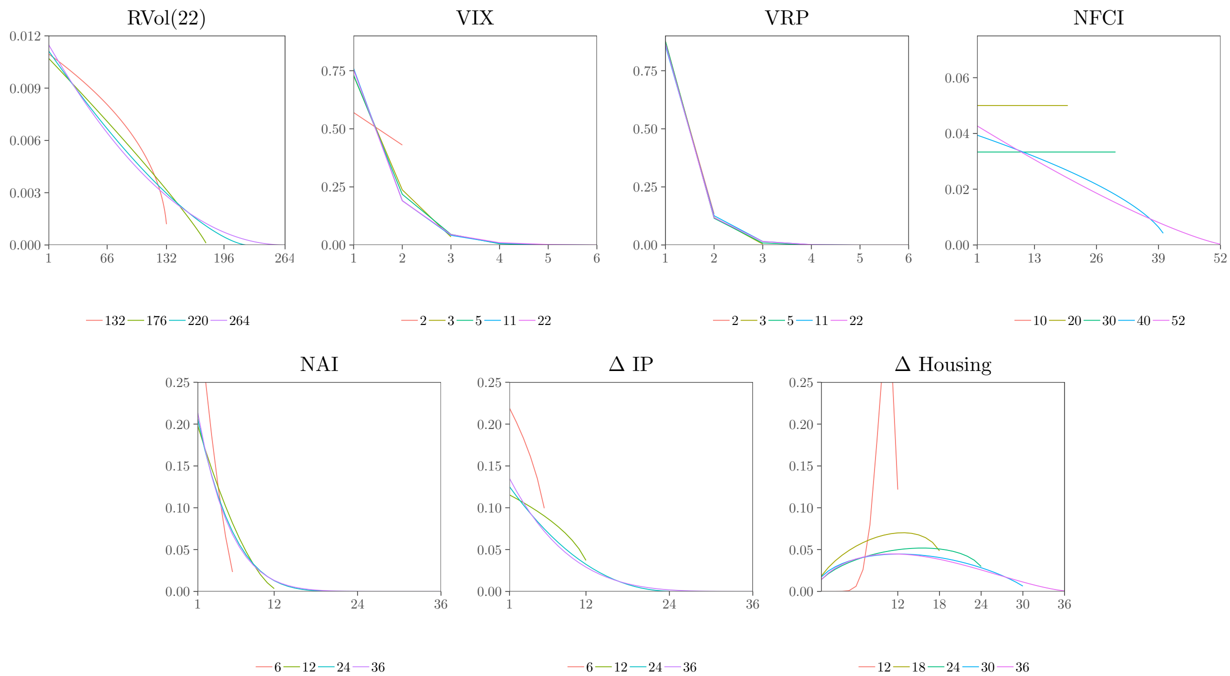
Figure C.2: Time series of explanatory variables.



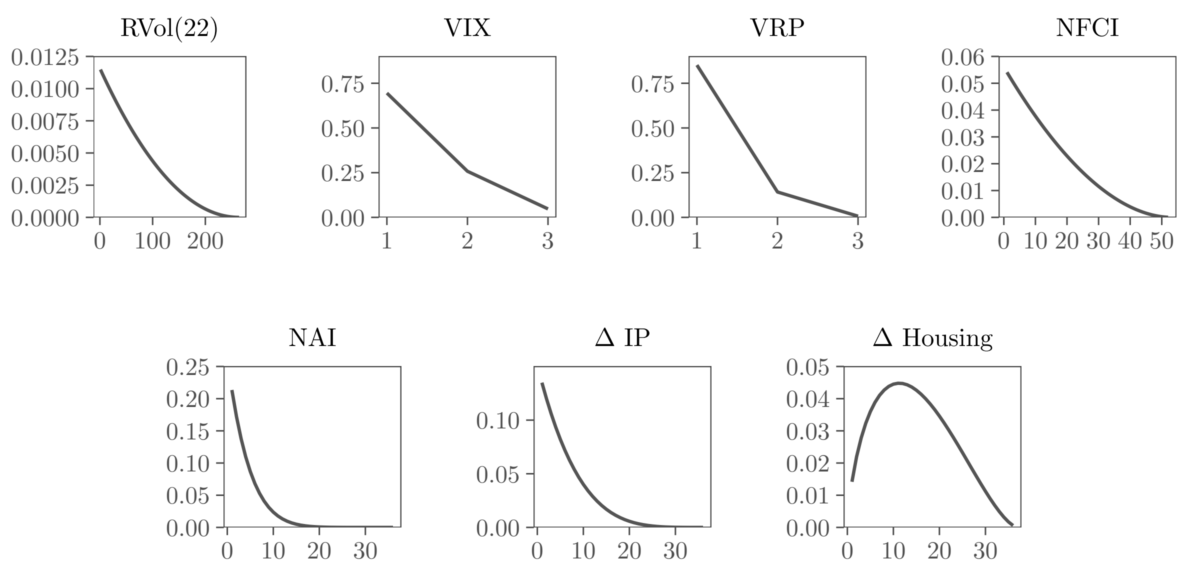
Notes: Daily financial data for the 1990:M1 to 2018:M4 period and macroeconomic data for the 1971:M1 to 2018:M4 period. See Section 4.1 for definitions and Table B.2 for descriptive statistics of those variables.

Figure C.3: Selected weighting schemes for different lag lengths.



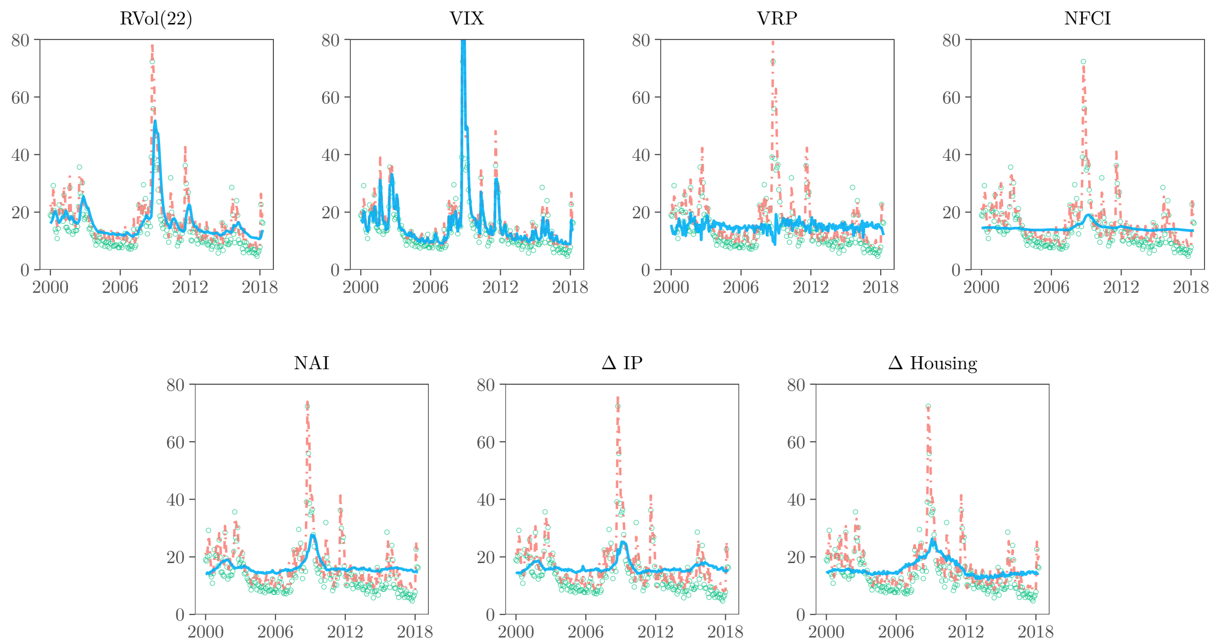
Notes: We depict selected weighting schemes that are implied by full-sample estimates for additional lag lengths K compared to those discussed in our empirical analysis.

Figure C.4: Weighting schemes for different explanatory variables.



Notes: For each explanatory variable, the estimated Beta weighting scheme (see eq. (9)) based on full sample estimates is depicted. For all variables except housing starts, we impose the restriction $w_1 = 1$. The corresponding parameters are reported in Table 4.

Figure C.5: Estimated monthly conditional volatility components.



Notes: The figure shows the monthly long-run volatility components $\sqrt{\tau_M}$ (blue, solid) and the monthly conditional volatilities $\sqrt{g_M \tau_M}$ (red, dot-dashed) for all GARCH-MIDAS models. To ensure comparability across the seven models, all figures cover the 2000:M1 to 2018:M4 period. Circles correspond to realized volatilities. Volatility is measured on an annualized scale.

D Simulations: Violation of Assumption 3

In the following we present the results of two additional simulations. The simulations cover scenarios in which Assumption 3 is violated. In this section, we consider daily explanatory variables (i.e. we set $I_t = 1$) because empirically a violation of Assumption 3 is more likely to occur for daily explanatory variables than for low-frequency explanatory variables. Both simulations show that even if Assumption 3 is violated, our theoretical results still apply.

First, we consider a daily explanatory variable, X_t, that is correlated with the daily innovations Z_t.[1] Recall that in our simulation the daily innovations are given by

$$Z_t = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_{n,t},$$

i.e. $Z_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. As before, we model X_t as an AR(1) process

$$X_t = \phi X_{t-1} + \xi_t$$

but the innovation is now given by

$$\xi_t / \sigma_\xi = \rho_{\xi,Z} Z_t + \sqrt{1 - \rho_{\xi,Z}^2}\tilde{\xi}_t,$$

where $\tilde{\xi}_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$, independent of Z_t and $\rho_{\xi,Z} \in [-1,1]$. In this setting, the correlation between the daily innovations Z_t and ξ_t is $\rho_{\xi,Z}$. We set $\rho_{\xi,Z} = -0.8$. The negative correlation between innovations to returns and innovations to X_t mimic the fact that changes in returns and daily measures of risk (such as the VIX index) are typically negatively correlated. Under our choice of $\phi = 0.98$, the contemporaneous correlation between Z_t and X_t is -0.16. Z_t is also correlated with future X_t but uncorrelated with past X_t.

In Table D.1, Panel A shows that on average the QML estimates are still close to the true parameter values and the asymptotic standard errors are accurate. Most importantly, Panel A of Figure D.1 illustrates that our results regarding the R^2 of a MZ regression still hold when X_t and Z_t are correlated. Panel A of Table D.2 shows the corresponding MCS inclusion rates. Clearly, the correctly specified GARCH-MIDAS model with $K = 264$ and the GARCH-MIDAS with misspecified lag-length still do very well. In contrast, for forecast horizons of up to two months the forecast performance of the MS-GARCH-TVI appears to deteriorate considerably.

Second, we consider the GARCH-MIDAS-RV model, i.e. we choose

$$X_t = RVol(22)_t = \sqrt{\frac{1}{22} \sum_{j=0}^{21} r_{t-j}^2}.$$

This choice corresponds to the GARCH-MIDAS-RV specification that is estimated in the empirical application in Section 4. Again, Z_t is correlated with the contemporaneous and future X_t but uncorrelated

[1] Since $I_t = 1$, we can drop the index i.

with lagged X_t. The results for this specification are presented in Panels B of Table D.1, Figure D.1 and Table D.2. Again, our previous findings regarding the Mincer-Zarnowitz R^2 and the MCS inclusion rates are confirmed.

Table D.1: Monte-Carlo parameter estimates: X_t and Z_t dependent.

	α	β	m	θ	w_2	$\kappa - 3$
Panel A: innovations to X_t correlated with Z_t						
GARCH-MIDAS (264)	0.000	-0.003	-0.001	0.008	0.890	-0.008
		{0.008}	{0.014}	{0.064}	{0.075}	{5.675}
		(0.008)	(0.014)	(0.063)	(0.075)	(7.741)
GARCH-MIDAS (66)	0.000	-0.003	0.002	-0.055	-3.185	-0.006
GARCH	0.003	0.003	0.034	—	—	0.017
Panel B: X_t given by $RVol(22)_t$						
GARCH-MIDAS (264)	-0.043	-0.034	0.370	-0.533	0.629	0.025
		{0.013}	{0.098}	{0.599}	{0.589}	{2.432}
		(0.013)	(0.079)	(0.329)	(0.321)	(4.555)
GARCH-MIDAS (66)	-0.045	-0.026	1.067	-1.230	1.823	0.032
GARCH	-0.052	0.087	1.373	—	—	0.048

Notes: Modified version of Panel A in Table 1 for the case of a daily varying long-term component but Assumption 3 being violated. In panel A, the true parameters are the same as in Table 1. However, the innovations ξ_t in the AR(1) process of X_t are correlated with Z_t, $\xi_t/\sigma_\xi = \rho_{\xi,Z} Z_t + \sqrt{1-\rho_{\xi,Z}^2}\tilde{\xi}_t, \tilde{\xi}_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. In Panel B, Assumption 3 is violated by employing a rolling window of past realized volatilities as a covariate, i.e. $X_t = RVol(22)_t = \sqrt{\frac{1}{22}\sum_{j=0}^{21} r_{t-j}^2}$. In this case, the GARCH-MIDAS parameters are given by $\mu = 0$, $\alpha = 0.1$, $\beta = 0.8$, $K = 264$, $m = -1$, $\theta = 1.6$, and $w_2 = 2.1$.

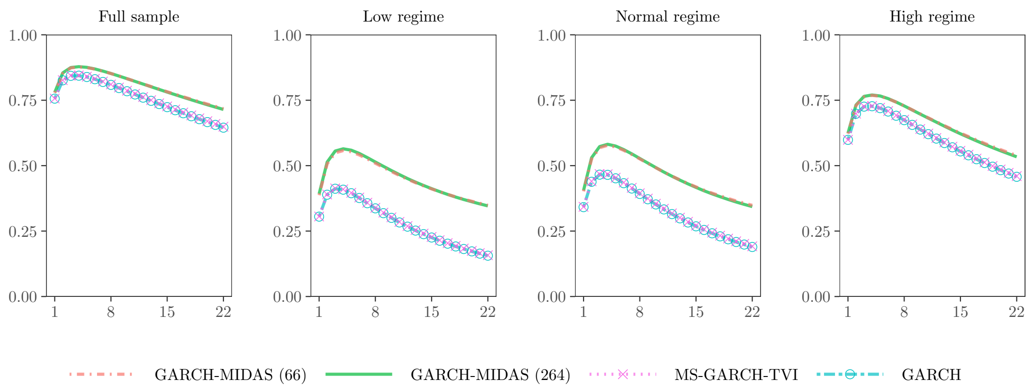
Table D.2: Model confidence set inclusion rates: X_t and Z_t dependent.

	1d	2w	1m	2m	3m
Panel A: innovations to X_t correlated with Z_t					
GARCH-MIDAS (264)	0.953	0.896	0.867	0.802	0.755
GARCH-MIDAS (66)	0.848	0.786	0.832	0.882	0.874
MS-GARCH-TVI	0.362	0.100	0.135	0.471	0.757
GARCH	0.259	0.038	0.048	0.251	0.496
Panel B: X_t given by $RVol(22)_t$					
GARCH-MIDAS (264)	0.932	0.892	0.887	0.878	0.857
GARCH-MIDAS (66)	0.371	0.140	0.097	0.197	0.301
MS-GARCH-TVI	0.743	0.654	0.640	0.757	0.827
GARCH	0.152	0.048	0.046	0.098	0.138

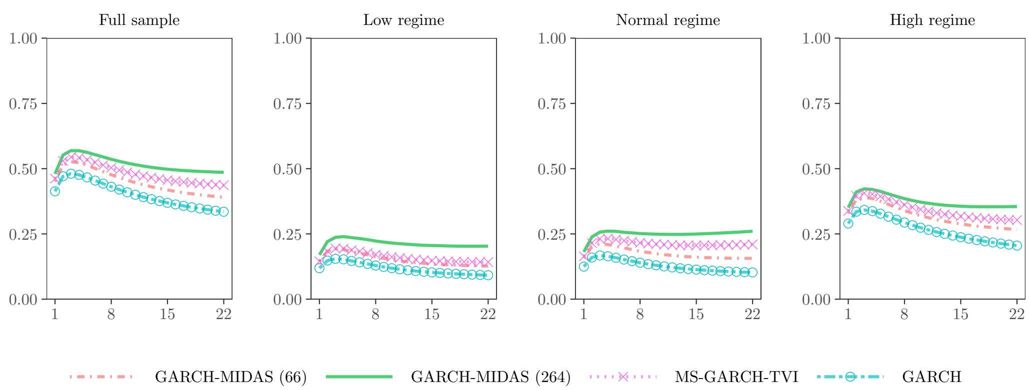
Notes: Modified version of the upper panel of Table 2 for two cases in which X_t depends on (past values of) Z_t. See notes of Table D.1 for a detailed description of these two scenarios.

Figure D.1: Mincer-Zarnowitz $R^2_{1:k}$—evaluation based on $RV_{1:k,t+1}$—X_t and Z_t dependent.

(a) innovations to X_t correlated with Z_t



(b) X_t given by $\mathit{RVol(22)}_t$



Notes: Modified version of Figure 5 for two scenarios in which X_t depends on (past values of) Z_t. See notes of Table D.1 for a detailed description of these two scenarios.

E Simulation with Diffusion Limit

In this section, we present simulation results for a situation in which the short-term discrete-time GARCH component (eq. (3)) has been replaced by its diffusion limit (see Nelson, 1990). In accordance with Andersen and Bollerslev (1998, pp. 894-895 and footnote 18 in the main text), we simulate the continuous-time data generating process using an Euler discretization scheme:

$$\varepsilon_{s+\Delta,t} = \ln P_{s+\Delta,t} - \ln P_{s,t} = \sqrt{\tau_t \tilde{g}_{s+\Delta,t} \Delta} W_{P,s,t}$$

with

$$\tilde{g}_{s+\Delta,t} = \tilde{\theta}\Delta + \tilde{g}_{s,t}\left(1 - \tilde{\theta}\Delta + \sqrt{2\tilde{\theta}\tilde{\lambda}\Delta} W_{\tilde{g},s,t}\right),$$

where $W_{P,s,t}$ and $W_{\tilde{g},s,t}$ are independent standard normal variables and the unit-variance GARCH-consistent parameters are given by

$$\tilde{\theta} = -\log(\alpha + \beta)$$

and

$$\tilde{\lambda} = 2\log(\alpha+\beta)^2 \cdot \Bigg\{ \left(\left(1 - (\alpha+\beta)^2\right) \cdot (1-\beta)^2 \right) \cdot \alpha^{-1} \cdot (1 - \beta \cdot (\alpha+\beta))^{-1} \Big) \\ + 6 \cdot \log(\alpha+\beta) + 2 \cdot \log(\alpha+\beta)^2 + 4 \cdot (1 - \alpha - \beta) \Bigg\}^{-1}.$$

We choose Δ such that we obtain 20 price changes per five-minute interval.

Tables E.1 and E.2 are the equivalent of Tables 1 and 2. Figures E.1 and E.2 are the equivalent of Figures 4 and 5.

As expected, the parameter estimates in Table E.1 are close to the ones in Table 1. Only in the case of a monthly τ_t do we observe an increase in bias for w_2. Moreover, we note that the excess kurtosis is considerably higher, even in comparison to our results regarding student-t distributed intraday returns. Figure E.1 makes it clear that we observe the same effect as in Figure 4. The same holds for Figure E.2 and the corresponding Figure 5 in the main text. Likewise, the MCS inclusion rates reported in Table E.2 confirm the overall results of Table 2 qualitatively. However, the MS-GARCH-TVI and GARCH models are less often excluded from the MCS.

Table E.1: Monte-Carlo parameter estimates with GARCH diffusion.

		α	β	m	θ	w_2	$\kappa - 3$
Monthly τ_t	GARCH-MIDAS (36)	-0.000	-0.007	-0.010	0.037	3.905	0.404
	GARCH-MIDAS (12)	-0.000	-0.006	-0.009	-0.029	0.396	0.406
	GARCH-MIDAS (36, $\tilde{X}$)	-0.000	-0.006	-0.009	-0.001	1.476	0.406
	GARCH-MIDAS (12, $\tilde{X}$)	-0.000	-0.005	-0.008	-0.076	-0.818	0.407
	GARCH	-0.000	0.001	0.005	—	—	0.421
Daily τ_t	M-GARCH (264)	-0.000	-0.006	-0.005	0.010	1.008	0.410
	GARCH-MIDAS (66)	-0.000	-0.005	-0.003	-0.050	-3.281	0.412
	GARCH-MIDAS (264, $\tilde{X}$)	-0.000	-0.006	-0.005	0.003	0.369	0.411
	GARCH-MIDAS (66, $\tilde{X}$)	0.000	-0.005	-0.002	-0.061	-3.448	0.414
	GARCH	0.003	0.001	0.030	—	—	0.442

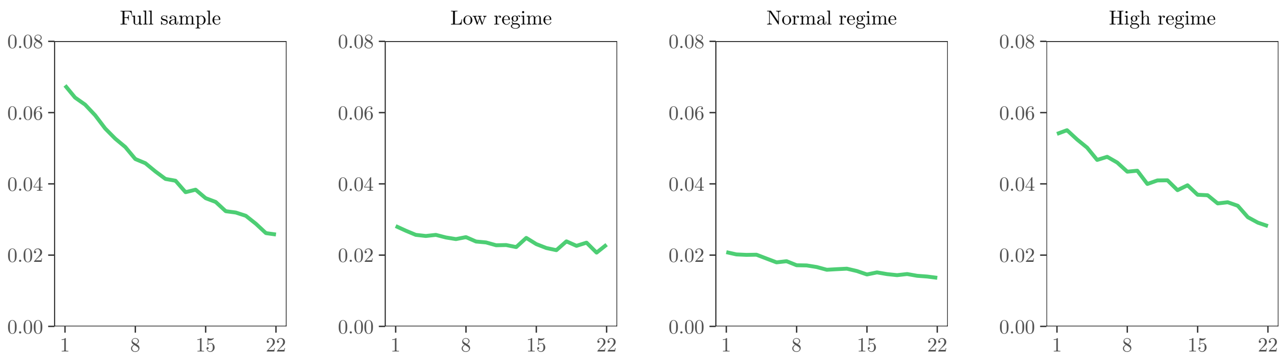
Notes: Modified version of the upper panel of Table 1. The only difference is that the short-term GARCH component is replaced by a consistent diffusion limit.

Table E.2: Model confidence set inclusion rates with GARCH diffusion.

		1d	2w	1m	2m	3m
Monthly τ_t	GARCH-MIDAS (36)	0.919	0.864	0.845	0.823	0.811
	GARCH-MIDAS (12)	0.918	0.873	0.854	0.846	0.837
	GARCH-MIDAS (36, $\tilde{X}$)	0.874	0.784	0.757	0.742	0.720
	M-GARCH (12, $\tilde{X}$)	0.852	0.784	0.746	0.734	0.715
	MS-GARCH-TVI	0.875	0.842	0.815	0.775	0.744
	GARCH	0.771	0.621	0.571	0.495	0.477
Daily τ_t	GARCH-MIDAS (264)	0.966	0.944	0.927	0.860	0.809
	GARCH-MIDAS (66)	0.935	0.915	0.916	0.907	0.880
	GARCH-MIDAS (264, $\tilde{X}$)	0.932	0.875	0.833	0.801	0.764
	GARCH-MIDAS (66, $\tilde{X}$)	0.905	0.860	0.841	0.848	0.855
	MS-GARCH-TVI	0.741	0.615	0.561	0.699	0.839
	GARCH	0.676	0.478	0.412	0.497	0.627

Notes: Modified version of the upper panel of Table 2. The only difference is that the short-term GARCH component is replaced by a consistent diffusion limit.

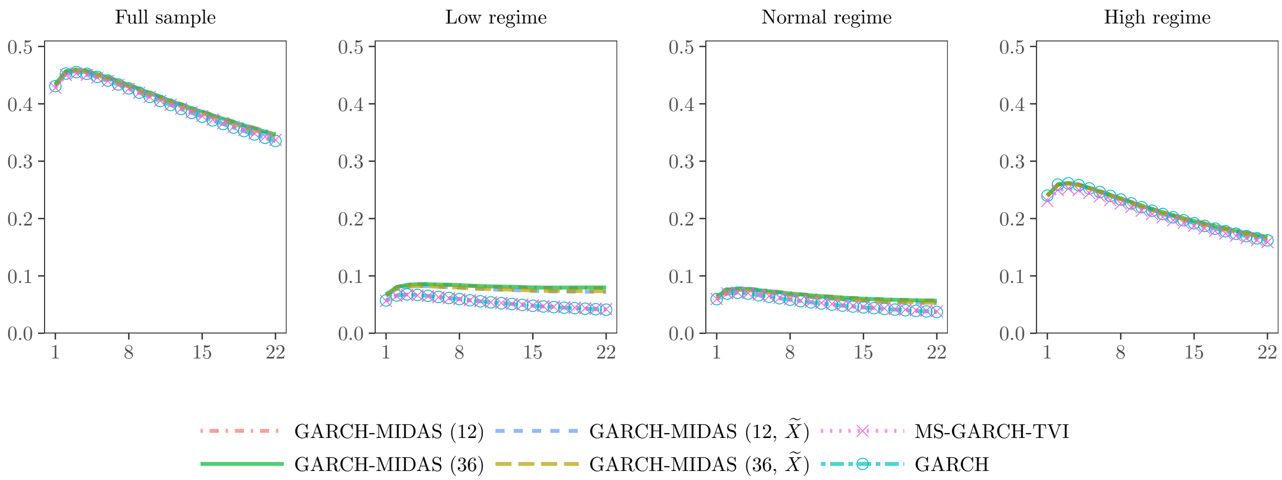
Figure E.1: Mincer-Zarnowitz R^2—monthly τ_t—evaluation based on $\varepsilon^2_{k,t+1}$ (with diffusion).



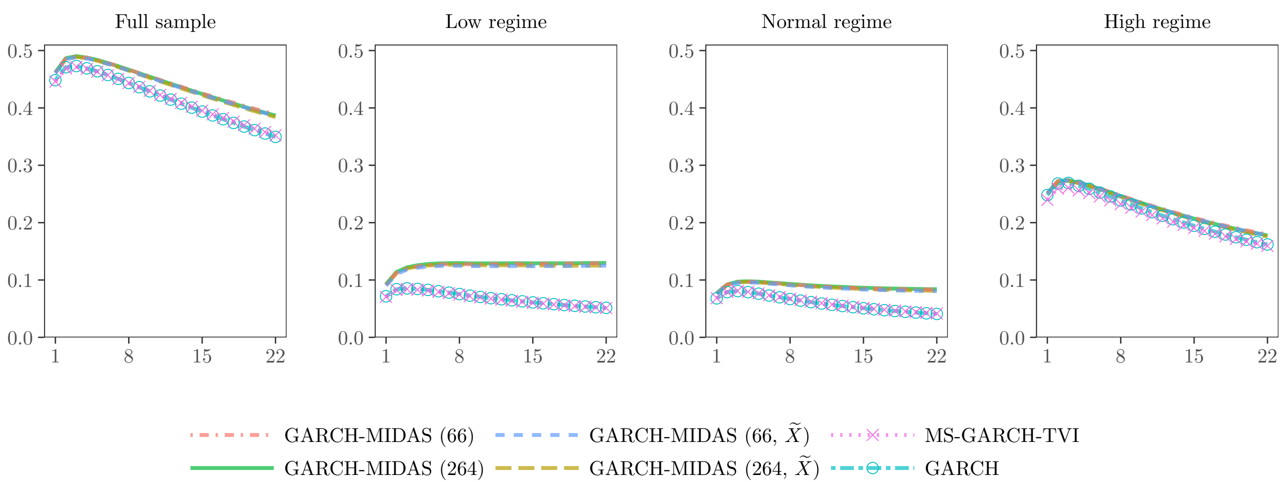
Notes: Modified version of Figure 4. The only difference is that the short-term GARCH component is replaced by a consistent diffusion limit.

Figure E.2: Mincer-Zarnowitz R^2—monthly and daily τ_t—evaluation based on $RV_{1:k,t+1}$ (with diffusion).

(a) Monthly τ_t



(b) Daily τ_t



Notes: Modified version of Figure 5. The only difference is that the short-term GARCH component is replaced by a consistent diffusion limit.

F Data

In this section, we provide detailed information on the data sources as well as on the data vintages that have been used. Whenever possible, we use real-time vintage data sets as available in ALFRED.[2] For downloading the respective data sources, we have written the R-package *alfred* (Kleen, 2017).[3] We make use of the following time series:

- Realized volatility based on five-minute intraday returns which are provided by the Realized Library of the Oxford-Man Institute of Quantitative Finance (Heber et al., 2009).

  http://realized.oxford-man.ox.ac.uk/data/download/

- The VIX index as a measure of option-implied volatility of S&P 500 returns (published by the Chicago Board Options Exchange).

  http://www.cboe.com/micro/vix/historical.aspx

- The Chicago Fed's National Financial Conditions Index (NFCI), measuring the risk, liquidity and leverage of money markets, debt and equity markets, and the traditional and shadow banking system. The NFCI takes positive/negative values whenever financial conditions are tighter/looser than on average.

  https://alfred.stlouisfed.org/series?seid=NFCI

- The Chicago Fed National Activity Index (NAI) is a weighted average of 85 filtered and standardized economic indicators. Whereas positive NAI values indicate an expanding US-economy above its historical trend rate, negative values indicate the opposite.

  https://alfred.stlouisfed.org/series?seid=CFNAI

- Industrial Production Index (IP), which is released by the Board of Governors of the Federal Reserve System.

  https://alfred.stlouisfed.org/series?seid=INDPRO

- New Privately Owned Housing Units Started (HOUST), which is published by the U.S. Bureau of the Census.

  https://alfred.stlouisfed.org/series?seid=HOUST

For the macroeconomic variables, we report the real-time data availability in Table F.1.

[2] https://alfred.stlouisfed.org
[3] https://cran.r-project.org/package=alfred

Table F.1: Real-time data availability.

Variable	Frequency	ALFRED ID	First Vintage Release
NFCI	weekly	NFCI	2011-05-25
NAI	monthly	CFNAI	2011-05-23
Industrial production	monthly	INDPRO	1973-12-14
Housing starts	monthly	HOUST	1973-12-18

Note: For each macroeconomic variable, we report the real-time data availability in the ALFRED data base.

G Description of Benchmark Models

For the empirical implementation, we use the statistical computing environment R (R Core Team, 2018). In the following, we present some details regarding the specification and estimation of the different models. For all benchmark models we have that $I_t = 1$ and, hence, the index i can be dropped.

- Two **Markov-Switching GARCH** models (MS-GARCH-TVI and MS-GARCH-TVC): Our specification follows Haas et al. (2004). Returns are decomposed as $\varepsilon_t = \tilde{\sigma}_{X_t,t} Z_t$, where $\{X_t\}$ is a Markov chain with a finite state space $S = \{1, 2\}$. The conditional variance in state $X_t = k$ is given by

$$\tilde{\sigma}^2_{k,t} = \omega_k + (\alpha_k + \gamma_k \mathbb{1}_{\{Z_{t-1}<0\}})\varepsilon^2_{t-1} + \beta_k \tilde{\sigma}^2_{k,t-1}.$$

  We employ two different specifications which nest the baseline GJR-GARCH model:

  1. An MS-GARCH called MS-GARCH-TVI (time-varying intercept) in which only the intercept is driven by the Markov chain while the ARCH/GARCH parameters are the same in both equations. In the simulations we set $\gamma_k = 0$.
  2. An MS-GARCH called MS-GARCH-TVC (time-varying coefficients) which models one regime as a GJR-GARCH and another regime as a standard GARCH(1,1), i.e. $\gamma_2 = 0$.[4]

  For estimation, we use the R-package *MSGARCH*, v2.3.[5] In both specifications we assume the innovations to be normally distributed which was numerically the most stable.

- As a generalization of the GARCH model, we employ the **Realized GARCH** model (Hansen et al., 2012). Here, the conditional variance of the returns $r_t - \mu^{RG} = \sqrt{\sigma_t^{RG}} Z_t^{RG}$, $Z_t^{RG} \overset{i.i.d.}{\sim} \mathcal{D}(0, 1)$ at day t is modeled as

$$\log \sigma_t^{RG} = \omega^{RG} + \alpha^{RG} \log RV_{t-1}^{int} + \beta^{RG} \log \sigma_{t-1}^{RG}$$

  and the realized measure RV_t^{int} as

$$\log RV_t^{int} = \xi^{RG} + \delta^{RG} \log \sigma_t^{RG} + \eta_1^{RG} Z_t^{RG} + \eta_2^{RG} \left(\left(Z_t^{RG}\right)^2 - 1 \right) + u_t^{RG}$$

  with $u_t^{RG} \overset{i.i.d.}{\sim} \mathcal{N}(0, \lambda^{RG})$. The innovations Z_t^{RG} and u_t^{RG} are independent. The estimation of the Realized GARCH model and the forecast computation by simulation is carried out using the R-package *rugarch*, v1.4-0 (Ghalanos, 2018).

- The **HEAVY** model by Shephard and Sheppard (2010) is a joint model of returns and some realized measure. We use the intraday realized variance, RV_t^{int}, as the realized measure. The conditional variance equation of daily returns is given by

$$\mathbf{Var}(\varepsilon_t^2 | \mathcal{F}_{t-1}) =: \sigma_t^{HVY} = \omega_1^{HVY} + \alpha_1^{HVY} RV_{t-1}^{int} + \beta_1^{HVY} \sigma_{t-1}^{HVY}$$

[4] Modeling both regimes as a GJR-GARCH turned out to be numerically unstable.

[5] Ardia, D., Bluteau, K., Boudt, K., Catania, L., Trottier, D.-A., 2017. Markov-Switching GARCH Models in R: The MSGARCH Package. *Journal of Statistical Software*, forthcoming.

and the realized measure equation by

$$\mathbf{E}[RV_t^{int}|\mathcal{F}_{t-1}] =: \sigma_t^{RV^{int}} = \omega_2^{HVY} + \alpha_2^{HVY} RV_{t-1}^{int} + \beta_2^{HVY} \sigma_{t-1}^{RV^{int}}.$$

We assume $\omega_1^{HVY}, \omega_2^{HVY}, \alpha_1^{HVY}, \alpha_2^{HVY}, \beta_2^{HVY} \geq 0$; $\beta_1^{HVY} \in [0, 1)$; and $\alpha_2^{HVY} + \beta_2^{HVY} \in [0, 1)$. The estimation is carried out by QML estimation. Note that both dynamic equations can be estimated separately. Often, the conditional variance equation is estimated to be unit-root. We compute iterative multi-step ahead forecasts, see Shephard and Sheppard (2010, eq. (11), p.205).

- We also consider a **HAR** specification that models realized variances directly (see Corsi, 2009). We specify the HAR model in terms of the log of the realized variances. The model for forecasting the k-period cumulative variance is given by

$$\log\left(\frac{RV_{t+1:t+k}}{k}\right) = b_0 + b_1 \log RV_t + b_2 \log\left(\frac{RV_{t-4:t}}{5}\right) + b_3 \log\left(\frac{RV_{t-21:t}}{22}\right) + \zeta_{t,k}$$

with $RV_{t+1:t+k} = \sum_{i=1}^{k} RV_{t+i}$. The HAR model is estimated by OLS. Realized variance forecasts are obtained as follows:

$$RV_{t+1:t+k|t} = k \cdot \exp\left(b_0 + b_1 \log RV_t + b_2 \log\left(\frac{RV_{t-4:t}}{5}\right) + b_3 \log\left(\frac{RV_{t-21:t}}{22}\right) + \frac{1}{2}\mathbf{Var}(\zeta_{t,k})\right),$$

assuming the residuals $\zeta_{t,k}$ to be normally distributed.

- **HAR with leverage** (Corsi and Reno, 2012):

$$\log\left(\frac{RV_{t+1:t+k}}{k}\right) = b_0^{lev} + b_1^{lev} \log RV_t + b_2^{lev} \log\left(\frac{RV_{t-4:t}}{5}\right) + b_3^{lev} \log\left(\frac{RV_{t-21:t}}{22}\right)$$
$$+ b_4^{lev} r_t + b_5^{lev} \times \frac{r_{t-4:t}}{5} + b_6^{lev} \times \frac{r_{t-21:t}}{22} + \zeta_{t,k}^{lev}$$

As in the case of the HAR model without leverage effect, we assume the residuals $\zeta_{t,k}^{lev}$ to be normally distributed in order to get closed-form expressions for the respective forecasts.

- The estimation of the **GARCH-MIDAS** models (see Section 2) has been carried out using QMLE, see Engle et al. (2013), and can be replicated using the R-package *mfGARCH*, v0.1.8, by Kleen (2018).[6]

[6] https://cran.r-project.org/package=mfGARCH

H Empirical Analysis: 95% Model Confidence Sets

As a robustness check, the following Tables H.1 and H.2 replicate Tables 6 and 7 for a confidence level of 95% instead of 90%.

Table H.1: QLIKE losses and 95% model confidence sets: full out-of-sample period.

	Full sample				
	1d	2w	1m	2m	3m
RVol(22)	0.306	0.246	0.271	0.387	0.428
VIX	0.275	0.215	0.240	0.359	0.414
VRP	0.291	0.227	0.260	0.384	0.430
NFCI	0.324	0.248	0.264	0.363	0.393
NAI	0.343	0.266	0.283	0.391	0.424
Δ IP	0.345	0.267	0.285	0.395	0.438
Δ Housing	0.328	0.252	0.264	**0.347**	**0.380**
VIX and NFCI	0.274	0.213	0.236	0.349	0.399
VIX and NAI	0.275	0.215	0.241	0.358	0.409
VIX and Δ IP	0.274	0.214	0.239	0.355	0.409
VIX and Δ Housing	0.275	0.218	0.243	0.351	0.405
Avg.	0.317	0.246	0.264	0.364	0.400
GARCH	0.342	0.263	0.282	0.395	0.434
MS-GARCH-TVI	0.362	0.292	0.315	0.426	0.488
MS-GARCH-TVC	0.355	0.271	0.283	0.387	0.421
RealGARCH	0.260	**0.206**	**0.233**	0.356	0.390
HEAVY	0.277	0.238	0.299	0.539	0.662
HAR	0.254	0.210	0.243	0.368	0.419
HAR (lev.)	**0.238**	0.207	0.245	0.371	0.419
No-change	0.358	0.498	0.636	1.157	1.292

Notes: See Table 6 but for a confidence level of 95% instead of 90%.

Table H.2: QLIKE losses and 95% model confidence sets: low/normal/high volatility regimes.

	low volatility regime					*normal volatility regime*					*high volatility regime*				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
RVol(22)	0.364	0.264	0.305	0.399	0.409	0.271	0.232	0.260	0.400	0.463	0.273	0.241	0.217	0.313	0.365
VIX	0.332	0.210	0.250	0.367	0.354	0.233	0.204	0.231	0.355	0.454	0.262	0.259	0.243	0.347	0.437
VRP	0.349	0.237	0.288	0.405	0.424	0.252	0.215	0.245	0.375	0.440	0.266	0.238	0.237	0.360	0.414
NFCI	0.400	0.274	0.304	0.389	0.402	0.272	0.228	0.248	0.364	0.417	0.292	0.244	0.217	0.293	0.297
NAI	0.438	0.308	0.338	0.432	0.460	0.284	0.240	0.260	0.384	0.427	0.292	0.241	0.216	0.309	0.322
Δ IP	0.441	0.313	0.343	0.437	0.468	0.287	0.241	0.262	0.389	0.447	0.288	**0.236**	**0.213**	0.310	0.335
Δ Housing	0.402	0.277	0.300	0.386	0.406	0.279	0.234	0.249	**0.331**	**0.385**	0.295	0.249	0.219	0.300	0.298
VIX and NFCI	0.331	0.212	0.250	0.364	**0.351**	0.235	**0.203**	**0.228**	0.345	0.440	**0.254**	0.249	0.229	0.321	0.391
VIX and Δ Indpro	0.332	0.211	0.251	0.363	0.351	0.234	0.204	0.230	0.354	0.450	0.257	0.253	0.237	0.338	0.424
VIX and NAI	0.333	0.212	0.254	0.367	0.354	0.234	0.205	0.231	0.358	0.449	0.259	0.254	0.237	0.334	0.417
VIX and Δ Housing	0.330	0.213	0.254	0.369	0.359	0.237	0.209	0.234	0.343	0.439	0.260	0.261	0.242	0.333	0.409
Avg.	0.396	0.273	0.303	0.391	0.403	0.269	0.226	0.247	0.362	0.418	0.272	0.240	0.217	0.306	0.335
GARCH	0.430	0.296	0.325	0.419	0.452	0.285	0.241	0.263	0.394	0.441	0.300	0.252	0.232	0.340	0.370
MS-GARCH-TVI	0.468	0.338	0.370	0.452	0.519	0.303	0.270	0.298	0.429	0.488	0.286	0.246	0.233	0.348	0.405
MS-GARCH-TVC	0.461	0.318	0.335	0.414	0.437	0.290	0.245	0.263	0.386	0.432	0.295	0.239	0.218	0.324	0.350
RealGARCH	0.237	**0.182**	**0.239**	0.380	0.409	0.256	0.208	0.229	0.358	0.408	0.331	0.261	0.229	**0.287**	**0.289**
HEAVY	0.272	0.223	0.326	0.591	0.759	0.262	0.228	0.273	0.498	0.593	0.339	0.305	0.313	0.535	0.642
HAR	0.234	0.189	0.254	**0.359**	0.385	0.243	0.212	0.238	0.374	0.430	0.340	0.257	0.230	0.371	0.470
HAR (lev.)	**0.226**	0.187	0.258	0.362	0.387	**0.232**	0.211	0.240	0.378	0.429	0.286	0.245	0.227	0.373	0.470
No-change	0.418	0.821	1.143	2.213	2.310	0.304	0.297	0.336	0.532	0.715	0.382	0.320	0.314	0.481	0.555

Notes: See Table 7 but for a confidence level of 95% instead of 90%.