Supplementary Appendix

A Unified Approach to Standardized-Residuals-Based Correlation Tests for GARCH-type Models

Yi-Ting Chen
Institute of Economics
Academia Sinica

In this supplementary appendix, we provide a graphical discussion on the PTS correlations and present additional simulation and empirical results that are not shown in the paper for the sake of space.

1 The PTS correlations

To interpret the possible dependence structures characterized by the PTS correlations, it is noted that $\rho_{ij,k}$ measures a linear association between v_t^i and v_{t-k}^j that may be understood as a deterministic relationship between v_t and v_{t-k}:

$$\frac{v_t^i-\zeta_i}{\sigma_i}=\rho_{ij,k}\frac{v_{t-k}^j-\zeta_j}{\sigma_j}, \tag{1}$$

where $\sigma_i^2 := \zeta_{2i}-\zeta_i^2$, because (1) exactly matches the definition of $\rho_{ij,k}$. (Recall that $\zeta_i := \mathbb{E}[v_t^i]$.) To further explore this relationship, we set $\zeta_4 = 3$ (kurtosis=3) and plot (1) for $\rho_{ij,k} = 0, \pm 0.5$, ± 1, and $i, j = 1, 2$ in Figure 1. The horizontal and vertical axes of these plots are, respectively, v_{t-k} and v_t. This figure shows that if $\rho_{ij,k} = 0$ (under the null of serial independence), (1) yields horizontal lines which indicate that v_t is not influenced by v_{t-k}. By contrast, if $\rho_{ij,k}$ is not equal to zero, then (1) generates curves that indicate the existence of certain linear or nonlinear relationships between v_t and v_{t-k} for which the slope and curvature (hence, the direction and strength of the dependence structures) are determined by the sign and magnitude of $\rho_{ij,k}$, respectively.

For the autocorrelation $\rho_{11,k}$, (1) is a straight line through the origin with the slope $\rho_{11,k}$. This structure is clearly consistent with the well-known concept of serial correlation. If $\rho_{11,k} > 0$ ($\rho_{11,k} < 0$), then v_t and v_{t-k} are positively (negatively) correlated. The size of $|\rho_{11,k}|$ determines the strength of this correlation. Conventionally, time series analysts use the AR or ARMA models to interpret such a linear relationship.

For the cross-correlation $\rho_{12,k}$, (1) is a parabola symmetric to $v_{t-k} = 0$ and concave to the horizontal axis when $\rho_{12,k} < 0$. This shape may indicate the dynamic structure in which the relationship between v_t and v_{t-k} is positively directive if $v_{t-k} < 0$ and negatively directive if $v_{t-k} > 0$. Conversely, in the case where $\rho_{12,k} > 0$, (1) is a parabola convex to the horizontal axis, which implies that v_t and v_{t-k} have a negative (positive) relationship when $v_{t-k} < 0$ ($v_{t-k} > 0$). As such, this cross-correlation is a sensible measure for regime-switching dynamics. In empirical studies, it is common to interpret such dynamics by using the SETAR model of Tong and Lim (1980), Teräsvirta's (1994) STAR (smooth transition AR) model, or other regime-switching models. Meanwhile, in the case where $\rho_{12,k} > 0$, (1) also shows that the level of v_t is positively correlated to the dispersion of v_{t-k}. Therefore, this cross-correlation may also be suitable for measuring the GARCH-in-mean effect of Engle et al. (1987).

For the cross-correlation $\rho_{21,k}$, (1) becomes a parabola symmetric to $v_t = 0$ and concave to the vertical axis when $\rho_{21,k} < 0$. This pattern indicates a negative relationship between the dispersion of v_t and the level of v_{t-k}. On the contrary, (1) in the case where $\rho_{21,k} > 0$ is a parabola symmetric to $v_t = 0$ and convex to the vertical axis, which implies a positive relationship between the dispersion of v_t and the level of v_{t-k}. The former is consistent with the leverage effect that is commonly observed in stock returns data; see e.g., Black (1976). Franses and van Dijk (2000, p.18) also provided some empirical evidence about the latter for exchange rate data. In both cases, there exists an asymmetric volatility effect. The EGARCH model, the GJR-GARCH model of Glosten et al. (1993), the asymmetric power ARCH model of Ding et al. (1993), and many other asymmetric GARCH-type models, are all designed to take this structure into consideration.

For the autocorrelation $\rho_{22,k}$, (1) generates an ellipse with the center $(v_{t-k}, v_t) = (0, 0)$ and a horizontal major axis when $\rho_{22,k} < 0$. This implies that the dispersion of v_t is negatively affected by that of v_{t-k}. This structure is interesting but relatively lacking in empirical relevance at least for original series. In comparison, if $\rho_{22,k} > 0$, (1) becomes a hyperbola reflecting the structure in which the dispersion of v_t is positively influenced by that of v_{t-k}. This is compatible with the volatility clustering effect, one of the most well-documented stylized facts of financial time series since Mandelbrot (1963). It is standard to interpret this effect by using Bollerslev's (1986) GARCH model.

This graphical discussion illustrates that $\rho_{11,k}$, $\rho_{12,k}$, $\rho_{21,k}$, and $\rho_{22,k}$ are, respectively, useful for measuring serial correlation, nonlinearity-in-mean (regime-switching dynamics or GARCH-in-mean), asymmetric volatility, and volatility clustering. As such, the $\rho_{ij,k}$-based C and Q tests may be applied to detecting the empirical relevance of these dependence structures. Moreover, since these structures are, respectively, the main data characteristics that the AR, STAR (GARCH-in-mean), GARCH, and EGARCH models aim to capture, this detection may provide constructive information for refining these popular time series models.

2 Simulation

To assess the robustness of our simulation results, we extend this simulation by considering another parameter set: $(a_0, a_1, a_2) = (0.05, 0.2, 0.6)$, $(b_0, b_1, b_2) = (0.01, 0.9, 0.05)$, $(a_{01}, a_{11}, a_{02}, a_{12}, a_3) = (0.05, 0.4, 0.05, -0.4, 0)$, $(b_0^e, b_1^e, b_2^e, b_3^e) = (-0.1, 0.9, -0.2, 0.05)$, and $(b_0', b_1', b_2', b_3') = (0.01, 0.6, 0.2, 0.19)$. Compared to the parameter set used in the paper, this set implies a higher strength of dependence structures. Corresponding to Tables 1 and 2 of the paper, we report the empirical size and power of existing tests and our tests under this new parameter set in Tables A.1 and A.2, respectively.

By comparing these tables, it is clearly seen that, as expected, the empirical power of our omnibus tests: Q_{PTS} and Q_{SCS} in Table A.2 is higher than that in Table 2. A similar result can also be found for the GS test. Moreover, under this new parameter set, the separate tests (our Q_{ij} tests and existing autocorrelation and LM tests) are also of higher power against the misspecifications that they are designed to detect. Nevertheless, our separate tests continue to outperform the existing autocorrelation and LM tests, and our omnibus tests perform in a way that is comparable to, or even better than, the GS test. The relative performance of our tests and existing tests discussed in Section 4 of the paper is robust to this change in parameters.

As discussed in the paper, the BHK test and the LM tests may not properly take into account the conditional mean estimation uncertainty. As suggested by a referee, it may be interesting to compare our tests with these existing tests in the case of no conditional mean. For this purpose, we consider the following DGPs:

(S1') the GARCH(1,1)-$N(0,1)$ process;

(S2') the GARCH(1,1)-$t(5)$ process;

(S3') the GARCH(1,1)-$L(0.5)$ process;

(P4') the EGARCH(1,1) process: $h_t = \exp(b_0^e + b_1^e \ln(h_{t-1}) + b_2^e v_{t-1} + b_3^e |v_{t-1}|^*)$;

(P5') the GARCH(1,2) process: $h_t = b_0' + b_1' h_{t-1} + b_2' u_{t-1}^2 + b_3' u_{t-2}^2$.

These DGPs are, respectively, the special cases of (S1), (S2), (S3), (P4), and (P5) where $\mu_t = 0$; see the paper for the latter. (The DGPs (P1), (P2), and (P3) are designed for assessing the power of tests against various types of conditional mean misspecifications. These DGPs degenerate to (S1') if $\mu_t = 0$.)

In Table A.3, we show the empirical size and power of the BHK and LM tests under (i) the parameter set used in the paper and (ii) the parameter set for Tables A.1 and

A.2. The empirical rejection frequencies of our tests under these two parameter sets are, respectively, shown in Tables A.4 and A.5. Again, by comparing these two tables, we can see that the relative performance of our tests and the BHK and LM tests in the case where $\mu_t = 0$ is the same as that in the case where $\mu_t \neq 0$ that we have discussed in the paper.

3 Empirical example

In the empirical example, the models being estimated and tested are of the following conditional mean and variance specifications:

(I) the AR(1)-GARCH(1,1) model:
$\mu_t = \theta_1 + \theta_2 y_{t-1}$ and $h_t = \theta_3 + \theta_4 h_{t-1} + \theta_5 u_{t-1}^2$;

(II) the AR(1)-EGARCH(1,1) model:
$\mu_t = \theta_1 + \theta_2 y_{t-1}$ and $h_t = \exp(\theta_3 + \theta_4 \ln(h_{t-1}) + \theta_5 v_{t-1} + \theta_6 |v_{t-1}|^*)$;

(III) the STAR(1,1)-GARCH(1,1) model:
$\mu_t = \mathbf{1}^*_{\{y_{t-1}<\theta_5\}}(\theta_1 + \theta_2 y_{t-1}) + \left[1 - \mathbf{1}^*_{\{y_{t-1}<\theta_5\}}\right](\theta_3 + \theta_4 y_{t-1})$
and $h_t = \theta_6 + \theta_7 h_{t-1} + \theta_8 u_{t-1}^2$;

(IV) the STAR(1,1)-EGARCH(1,1) model:
$\mu_t = \mathbf{1}^*_{\{y_{t-1}<\theta_5\}}(\theta_1 + \theta_2 y_{t-1}) + \left[1 - \mathbf{1}^*_{\{y_{t-1}<\theta_5\}}\right](\theta_3 + \theta_4 y_{t-1})$
and $h_t = \exp(\theta_6 + \theta_7 \ln(h_{t-1}) + \theta_8 v_{t-1} + \theta_9 |v_{t-1}|^*)$;

(V) the AR(1)-GARCH(1,1) model:
$\mu_t = \theta_1 + \theta_2 y_{t-1} + \theta_3 \ln(h_t)$ and $h_t = \theta_4 + \theta_5 h_{t-1} + \theta_6 u_{t-1}^2$;

(VI) the AR(1)-EGARCH(1,1) model:
$\mu_t = \theta_1 + \theta_2 y_{t-1} + \theta_3 \ln(h_t)$ and $h_t = \exp(\theta_4 + \theta_5 \ln(h_{t-1}) + \theta_6 v_{t-1} + \theta_7 |v_{t-1}|^*)$.

In Table A.5, we show the Gaussian QMLEs of these models obtained by the Broyden-Fletcher-Goldfarb-Shanno method. Meanwhile, we also report the Kiefer-Salmon (1983) test statistic:

$$KS = T\left(\frac{(\hat{m}_3 - 3\hat{m}_1)^2}{6} + \frac{(\hat{m}_4 - 6\hat{m}_2 + 3)^2}{24}\right),$$

where $\hat{m}_k := T^{-1}\sum_{t=1}^{T} \hat{v}_t^k$ denotes the k-th sample moment of the $\hat{v}_t$'s, for checking the hypothesis of conditional normality. This test statistic is of the asymptotic null distribution $\chi^2(2)$. Bontemps and Meddahi (2005) proved that the Kiefer-Salmon test is robust to the effect of estimation uncertainty and hence applicable to the standardized residuals of GARCH-type models.

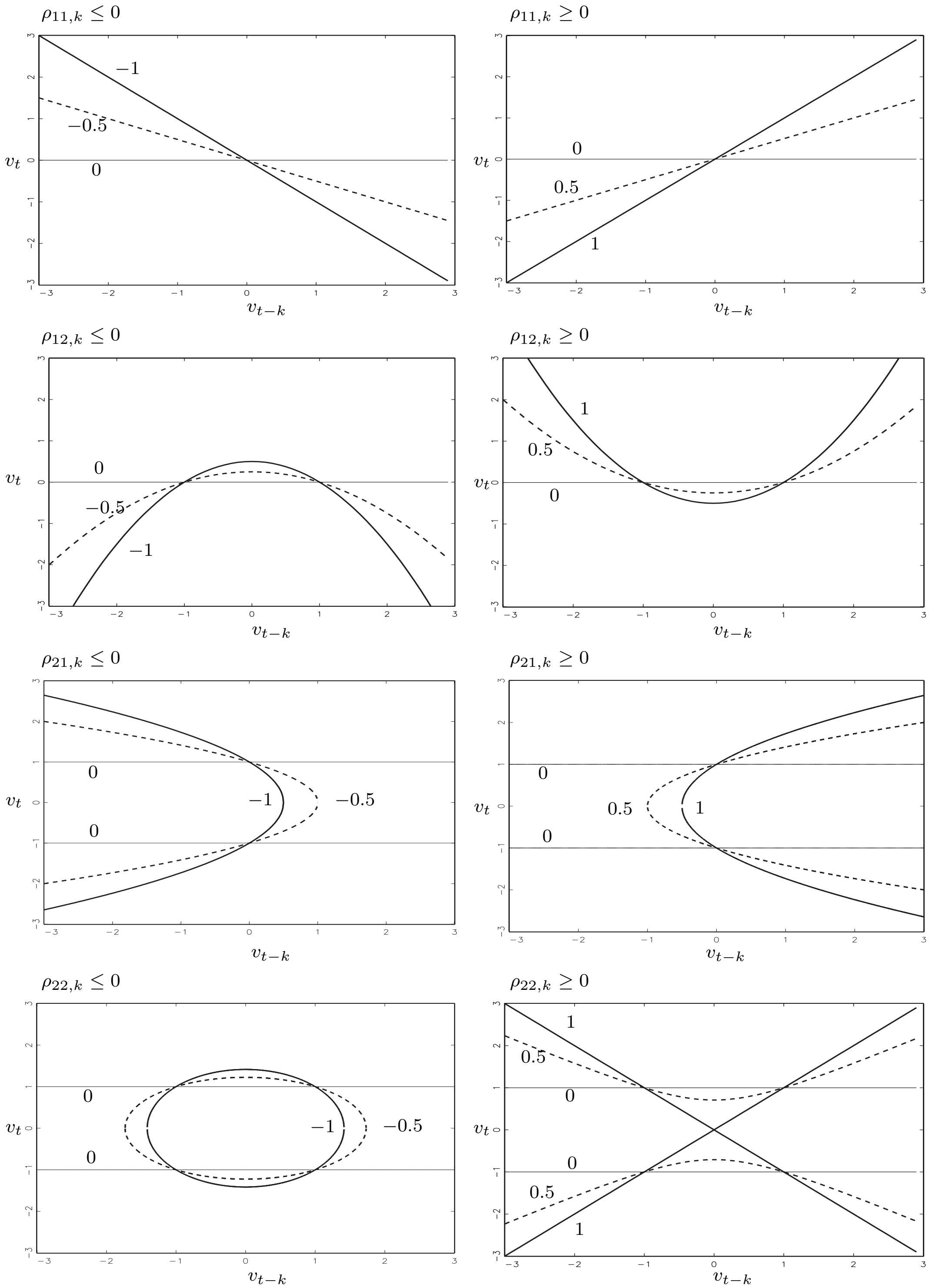

Figure 1: Possible deterministic relationships between v_t and v_{t-k} for the $\rho_{ij,k}$'s with $i, j = 1, 2$.

Table A.1: Empirical size and power: Existing tests.

		The autocorrelation tests													
		$T = 500$				$T = 1000$						The LM and GS tests			
DGP		$k = 1$	$k = 3$	$k = 5$	$m = 5$	$k = 1$	$k = 3$	$k = 5$	$m = 5$			(I)	(II)	(III)	(IV)
size	Q_{BP}	0.0	5.1	3.4	2.0	0.0	6.0	4.8	3.3	T =500	LM	14.7	8.8	8.7	11.3
	Q_{ML}	2.7	4.2	3.0	3.3	2.2	4.2	5.5	4.7		GS	1.7	2.2	2.3	2.5
(S1)	Q_{LM}	6.0	6.0	3.6	9.2	5.0	5.5	6.4	8.2	T =1000	LM	12.0	8.0	7.6	7.3
	Q_{BHK}	5.5	5.7	3.6	6.1	4.4	5.5	6.4	7.3		GS	2.7	3.0	2.8	2.9
	Q_{BP}	0.1	5.0	5.6	1.9	0.0	4.3	4.7	2.2	T =500	LM	7.3	5.4	7.1	5.5
	Q_{ML}	0.6	2.5	3.4	3.0	1.2	3.2	3.7	3.4		GS	3.8	4.1	4.3	4.3
(S2)	Q_{LM}	9.0	4.6	4.2	3.8	10.4	4.8	4.8	4.6	T =1000	LM	5.4	6.2	7.2	5.6
	Q_{BHK}	1.3	3.0	3.6	4.2	2.8	3.8	3.9	4.4		GS	3.9	3.5	4.1	4.0
	Q_{BP}	0.1	3.3	5.6	1.5	0.0	6.2	4.7	2.6	T =500	LM	6.5	6.4	14.1	10.9
	Q_{ML}	1.1	2.5	2.9	2.9	1.0	2.7	3.3	3.2		GS	7.3	7.2	6.8	7.3
(S3)	Q_{LM}	9.1	3.8	3.9	3.8	11.0	5.8	4.7	3.0	T =1000	LM	4.5	4.5	10.7	8.2
	Q_{BHK}	1.8	2.8	3.2	4.2	1.8	3.0	3.6	3.7		GS	6.2	6.5	6.7	7.1
power	Q_{BP}	100.0	20.7	7.9	100.0	100.0	32.3	7.1	100.0	T =500	LM	14.1	6.3	6.9	8.3
	Q_{ML}	2.1	2.7	3.3	6.2	1.5	3.7	4.4	11.3		GS	99.9	100.0	100.0	100.0
(P1)	Q_{LM}	4.5	4.5	3.9	10.2	5.3	5.9	5.1	18.3	T =1000	LM	20.1	6.7	6.6	6.2
	Q_{BHK}	4.7	4.4	3.8	9.4	4.9	5.6	5.3	17.8		GS	100.0	100.0	100.0	100.0
	Q_{BP}	0.0	5.4	3.1	1.7	0.0	6.3	6.3	3.8	T =500	LM	17.0	12.5	9.2	14.5
	Q_{ML}	5.8	3.4	3.6	3.6	12.3	3.2	3.6	6.9		GS	92.4	91.1	89.5	87.8
(P2)	Q_{LM}	13.0	4.5	4.1	8.4	22.1	5.0	4.1	15.6	T =1000	LM	19.3	17.5	9.3	20.9
	Q_{BHK}	11.7	4.2	4.1	9.4	21.1	5.0	4.0	15.8		GS	100.0	100.0	100.0	99.9
	Q_{BP}	0.3	26.0	27.1	43.3	0.1	38.7	39.0	64.0	T =500	LM	26.2	10.5	9.2	13.8
	Q_{ML}	2.4	4.4	4.8	4.0	2.8	5.2	4.1	4.9		GS	18.1	22.8	26.7	29.7
(P3)	Q_{LM}	5.7	5.5	5.1	9.7	6.5	6.4	5.4	11.9	T =1000	LM	21.8	11.4	11.0	10.8
	Q_{BHK}	5.0	5.2	5.1	6.5	6.1	6.4	5.3	9.2		GS	43.2	52.3	57.9	63.7
	Q_{BP}	0.0	5.2	4.0	1.9	0.0	4.8	4.9	2.6	T =500	LM	6.7	76.8	71.7	71.6
	Q_{ML}	0.4	2.7	3.5	2.1	0.8	2.4	2.7	1.3		GS	37.9	40.8	41.4	42.4
(P4)	Q_{LM}	6.7	3.8	4.5	6.8	7.4	3.5	3.6	8.1	T =1000	LM	4.3	96.0	93.4	95.6
	Q_{BHK}	4.4	3.7	4.2	4.2	4.2	3.3	3.5	3.3		GS	80.6	84.1	86.2	87.1
	Q_{BP}	0.0	3.8	5.1	3.0	0.0	5.7	4.2	3.3	T =500	LM	35.1	2.2	6.5	17.5
	Q_{ML}	15.3	8.4	5.8	19.3	42.9	9.2	6.5	41.4		GS	3.4	4.0	4.2	4.5
(P5)	Q_{LM}	36.6	10.1	5.9	27.5	71.9	11.7	7.0	48.1	T =1000	LM	52.4	1.9	9.2	26.9
	Q_{BHK}	33.6	9.7	5.9	23.9	70.2	11.6	7.0	46.8		GS	8.0	8.0	9.4	10.8

Notes: The entries are rejection frequencies in percentages. The notations "Q_{BP}", "Q_{ML}", "Q_{LM}", and "Q_{BHK}" represent the Box-Pierce, McLeod-Li, Li-Mak, and BHK tests, respectively. For these autocorrelation tests, Columns "$k = 1$", "$k = 3$", and "$k = 5$", respectively, correspond to the special cases where, given $m = 1$, $k = 1$, $k = 3$, and $k = 5$; the significance of these test statistics is evaluated using the distribution $\chi^2(1)$. In the case of $m = 5$, the significance of the autocorrelation test statistics is evaluated by using the distribution $\chi^2(5)$. For the LM test (the GS test), Columns (I), (II), (III), and (IV), respectively, correspond to the misspecification indicators: $\eta_t = \eta_t^V$, η_t^N, η_t^{NSB}, and η_t^{JB} (the preliminary bandwidths: $\bar{p} = 6, 9, 12$, and 15).

Table A.2: Empirical size and power: Our tests.

		The PTS correlation tests									The SCS correlation tests							
		$T = 500$				$T = 1000$					$T = 500$				$T = 1000$			
DGP		$C(1)$	$C(3)$	$C(5)$	$Q(5)$	$C(1)$	$C(3)$	$C(5)$	$Q(5)$		$C(1)$	$C(3)$	$C(5)$	$Q(5)$	$C(1)$	$C(3)$	$C(5)$	$Q(5)$
size	11	4.9	4.8	3.4	5.7	5.6	6.1	4.6	7.2	ss	4.2	4.5	4.8	5.4	5.7	5.8	3.9	5.9
	12	5.0	4.1	4.8	4.2	5.8	4.4	4.0	3.4	sc	4.7	5.1	5.3	5.2	4.4	4.5	4.7	3.8
(S1)	21	4.9	4.2	5.3	4.1	5.1	5.7	4.3	4.5	cs	4.6	4.1	5.7	4.5	4.8	5.3	3.9	4.8
	22	5.4	5.7	4.2	6.1	5.7	5.9	5.6	6.1	cc	4.6	5.9	3.6	4.4	4.7	5.4	6.6	5.7
	PTS	12.7	11.3	7.7	14.0	10.3	9.1	5.2	9.5	SCS	6.0	5.3	5.9	5.9	7.6	6.0	5.1	5.5
	11	4.1	5.3	5.5	3.6	4.7	3.9	4.3	5.4	ss	4.5	6.4	4.6	5.6	5.7	2.9	5.7	5.1
	12	3.8	3.6	3.8	2.9	4.6	3.8	4.1	3.7	sc	4.9	4.6	5.8	4.3	4.8	4.3	5.0	4.0
(S2)	21	4.0	5.3	4.3	3.3	4.6	4.0	3.6	3.6	cs	5.5	4.5	5.2	5.2	4.8	4.9	4.8	3.9
	22	6.1	5.4	5.6	7.0	6.7	6.2	6.0	9.2	cc	4.1	5.8	4.8	5.8	5.6	5.2	5.7	6.3
	PTS	8.7	8.9	8.2	10.8	8.5	8.5	7.7	8.2	SCS	5.6	5.9	6.2	6.0	5.3	4.5	5.2	4.6
	11	5.6	4.2	4.9	6.0	4.3	6.5	4.9	5.9	ss	6.1	4.5	5.4	6.1	5.9	5.9	5.4	5.2
	12	4.3	5.5	4.8	4.1	6.3	6.2	5.3	5.1	sc	4.7	4.4	5.2	4.1	4.8	4.4	4.2	4.0
(S3)	21	6.6	4.7	5.6	7.5	6.8	4.7	5.4	5.6	cs	5.2	5.6	4.9	4.9	6.0	5.1	4.8	5.7
	22	5.9	7.0	7.2	6.8	7.1	9.3	8.3	10.0	cc	5.1	5.5	4.6	6.0	5.5	5.4	6.4	5.7
	PTS	9.8	9.3	6.1	14.7	8.7	9.3	7.0	11.1	SCS	8.0	5.5	4.8	10.0	6.6	5.5	4.2	6.3
power	11	100.0	22.5	8.0	100.0	100.0	34.7	7.2	100.0	ss	99.7	20.4	7.5	100.0	95.9	33.3	7.2	100.0
	12	3.0	5.8	5.9	3.0	2.9	5.9	4.6	4.1	sc	3.3	4.9	3.9	3.5	5.2	5.9	4.7	4.1
(P1)	21	3.4	3.9	5.4	3.8	3.4	6.2	4.3	3.5	cs	4.4	4.3	5.0	4.4	3.3	4.9	4.8	3.5
	22	5.1	8.3	7.7	11.8	5.7	9.9	7.8	18.6	cc	5.5	6.2	6.3	10.3	6.0	8.4	6.7	16.9
	PTS	100.0	19.8	10.2	100.0	100.0	28.8	8.5	100.0	SCS	96.5	14.6	8.1	99.2	84.5	23.2	6.7	100.0
	11	7.2	5.9	3.3	6.3	5.9	6.8	5.9	7.6	ss	7.0	5.6	5.1	6.5	6.2	7.6	5.4	6.2
	12	99.5	5.5	4.9	93.5	100.0	7.5	5.2	100.0	sc	99.6	6.0	5.2	95.7	91.2	7.2	4.4	100.0
(P2)	21	4.8	5.5	4.6	3.6	6.9	4.4	4.6	5.6	cs	4.4	5.8	4.1	4.7	5.1	3.9	5.6	5.2
	22	5.7	6.9	6.3	10.7	11.8	7.4	6.5	13.6	cc	4.4	5.7	5.5	7.4	5.1	6.6	6.5	10.9
	PTS	97.7	12.0	6.7	82.7	99.9	13.3	9.5	98.8	SCS	95.8	7.7	5.0	80.1	76.9	7.0	6.4	98.7
	11	19.9	27.3	27.1	55.7	30.0	39.2	39.4	72.9	ss	15.2	17.5	19.7	37.3	7.0	30.4	30.0	54.3
	12	40.4	31.7	25.2	63.1	68.2	54.9	46.7	96.7	sc	31.2	24.7	20.7	54.8	44.1	43.9	38.0	90.9
(P3)	21	4.8	5.9	5.4	5.1	4.8	5.7	6.7	6.6	cs	5.2	5.4	4.5	4.6	6.2	6.4	6.4	6.0
	22	5.5	4.2	4.6	6.3	4.6	4.5	4.1	6.3	cc	5.4	6.4	4.7	5.4	6.5	4.0	5.0	5.8
	PTS	48.0	39.8	28.8	76.8	70.5	58.0	50.3	97.4	SCS	30.7	25.7	20.4	58.7	31.8	43.7	38.2	89.3
	11	7.3	5.3	4.0	5.4	12.8	4.7	5.0	7.6	ss	4.5	6.3	4.7	5.5	4.5	4.1	5.6	4.9
	12	4.4	5.0	5.2	4.0	4.0	6.5	4.3	4.4	sc	4.2	5.5	5.5	4.6	4.8	5.1	5.2	4.9
(P4)	21	75.2	43.1	24.6	95.3	96.2	71.4	42.0	100.0	cs	69.4	37.9	21.0	91.0	46.2	62.9	38.2	100.0
	22	6.6	5.1	5.0	6.5	5.9	4.2	4.1	6.7	cc	4.4	4.3	4.3	5.7	4.1	4.4	4.0	4.2
	PTS	57.1	29.3	16.7	68.3	89.6	54.3	29.7	99.5	SCS	48.8	24.0	12.4	64.5	28.6	42.4	21.5	98.1
	11	5.3	3.3	4.2	4.7	5.3	5.5	4.1	5.3	ss	5.7	4.1	4.4	6.3	6.4	4.6	4.8	5.5
	12	5.6	4.8	4.6	4.6	4.7	4.9	5.6	4.3	sc	5.5	5.1	4.8	4.7	4.8	4.9	5.5	4.8
(P5)	21	3.7	4.9	4.6	3.8	3.1	4.8	4.7	3.6	cs	3.9	5.1	5.3	5.2	3.9	4.7	4.8	4.2
	22	47.3	6.3	5.3	33.4	75.8	6.8	5.9	61.0	cc	43.9	8.0	6.4	35.0	74.1	9.1	7.4	65.4
	PTS	35.1	5.0	6.2	12.2	60.9	4.7	5.4	23.9	SCS	29.0	5.4	5.5	12.7	56.4	5.6	5.2	26.8

Notes: The entries are rejection frequencies in percentages. For the PTS (SCS) correlation tests, Column "$Q(5)$" and Rows "11", "12", "21", and "22", "PTS" ("ss", "sc", "cs", and "cc", "SCS"), respectively, correspond to the $Q_{11}, Q_{12}, Q_{21}, Q_{22}$, and Q_{PTS} ($Q_{ss}, Q_{sc}, Q_{cs}, Q_{cc}$, and Q_{SCS}) tests with $m = 5$. Columns "$C(1)$", "$C(3)$", and "$C(5)$", respectively, correspond to the special cases of these Q tests, where, given $m = 1$, $k = 1, 3$, and 5; that is, the C tests.

Table A.3: Empirical size and power: The BHK and LM tests ($\mu_t = 0$)

DGP	set	T	The BHK test $k=1$	$k=3$	$k=5$	$m=5$	The LM tests (I)	(II)	(III)	(IV)
size	(i)	500	5.4	4.6	5.0	5.7	17.0	5.2	4.8	7.9
(S1')		1000	6.2	5.5	5.0	5.8	12.1	5.4	4.6	6.6
	(ii)	500	6.7	4.8	5.0	6.8	18.1	5.4	5.6	7.7
		1000	5.6	3.6	5.0	6.5	11.3	5.9	4.1	7.2
	(i)	500	2.2	3.3	2.7	5.4	9.8	2.4	3.8	4.3
(S2')		1000	3.6	3.7	3.3	5.2	6.6	3.1	3.1	4.6
	(ii)	500	2.8	3.0	3.0	2.9	6.5	2.4	4.0	4.0
		1000	3.2	3.5	2.4	4.6	6.0	1.9	3.3	4.1
	(i)	500	2.6	2.3	2.8	3.2	5.3	1.8	6.3	4.5
(S3')		1000	2.1	2.2	3.7	5.6	6.4	2.7	5.8	5.3
	(ii)	500	1.5	2.0	3.4	3.8	7.0	2.9	6.7	4.8
		1000	3.6	2.0	3.7	4.9	6.6	2.8	6.6	5.9
	(i)	500	5.3	3.7	4.3	9.0	15.7	19.5	23.2	18.6
(P4')		1000	4.0	5.2	3.4	4.4	8.2	39.5	41.1	35.7
	(ii)	500	3.9	4.0	4.3	4.4	7.0	59.3	61.8	53.2
		1000	4.3	5.0	4.7	4.1	4.9	91.5	90.7	89.5
	(i)	500	25.4	7.2	4.9	15.5	22.1	1.8	5.8	11.5
(P5')		1000	48.2	10.4	5.0	30.5	32.4	1.8	8.6	21.6
	(ii)	500	33.6	8.7	6.2	25.7	33.5	1.3	5.7	12.6
		1000	65.7	11.0	6.4	46.8	52.0	1.7	8.9	21.0

Notes: The entries are rejection frequencies in percentages. For the BHK test, Columns "$k = 1$", "$k = 3$", and "$k = 5$", respectively, correspond to the special cases where, given $m = 1$, $k = 1$, $k = 3$, and $k = 5$; the significance of these test statistics is evaluated using the distribution $\chi^2(1)$. For the LM test (the GS test), Columns (I), (II), (III), and (IV), respectively, correspond to the misspecification indicators: $\eta_t = \eta_t^V$, η_t^N, η_t^{NSB}, and η_t^{JB}. Sets (i) and (ii) represent the parameter sets discussed in Section 2 of this appendix.

Table A.4: Empirical size and power: Our tests ($\mu_t = 0$).

	DGP		The PTS correlation tests									The SCS correlation tests							
			$T = 500$				$T = 1000$					$T = 500$				$T = 1000$			
			$C(1)$	$C(3)$	$C(5)$	$Q(5)$	$C(1)$	$C(3)$	$C(5)$	$Q(5)$		$C(1)$	$C(3)$	$C(5)$	$Q(5)$	$C(1)$	$C(3)$	$C(5)$	$Q(5)$
	size	11	5.2	5.0	4.2	4.8	5.1	5.1	5.3	3.8	ss	5.1	4.7	5.1	4.9	5.3	5.6	4.8	4.9
		12	4.4	4.3	3.8	3.3	4.9	6.0	6.8	4.9	sc	4.3	4.4	3.8	3.7	5.8	5.5	6.7	5.2
	(S1')	21	4.8	5.6	4.7	4.2	5.5	4.9	4.9	4.9	cs	5.7	4.8	5.0	5.0	4.9	4.9	4.3	4.8
		22	7.3	7.0	7.2	9.3	6.7	6.9	5.6	7.3	cc	5.6	6.2	6.5	7.2	6.2	6.1	5.0	5.8
		PTS	11.5	7.4	7.7	13.5	10.2	8.7	6.8	10.2	SCS	5.9	5.2	5.5	8.3	7.2	6.6	5.1	6.0
		11	5.2	5.4	4.5	5.1	5.3	6.1	5.1	6.1	ss	4.9	4.7	4.2	6.1	5.7	4.8	4.8	4.0
		12	2.4	3.7	3.9	3.0	3.9	4.2	4.7	5.2	sc	4.1	6.3	4.8	5.0	4.8	4.9	4.5	5.2
	(S2')	21	3.4	4.0	4.9	4.1	3.5	3.7	4.7	2.4	cs	3.5	5.9	6.1	6.6	5.8	4.9	5.3	5.1
		22	5.8	7.6	6.9	7.6	8.2	8.3	8.6	10.9	cc	5.5	7.2	4.0	4.5	6.0	5.4	5.4	5.7
		PTS	9.4	9.5	8.3	9.7	9.4	10.2	10.6	9.9	SCS	5.4	7.5	6.4	6.1	6.5	6.1	6.0	7.5
		11	6.1	5.3	5.2	5.1	4.8	4.9	4.4	4.5	ss	4.4	4.4	4.6	4.0	4.6	4.5	4.6	4.8
		12	6.1	5.5	4.7	4.4	5.5	5.9	5.8	4.9	sc	5.4	5.8	5.0	4.8	5.7	4.0	4.9	5.1
(i)	(S3')	21	4.3	5.6	6.0	4.9	5.0	5.2	5.6	6.1	cs	4.6	5.2	4.5	4.7	3.6	5.1	6.1	5.2
		22	6.0	8.3	8.4	7.7	7.5	8.3	10.6	10.4	cc	5.2	5.1	5.7	6.2	5.6	4.7	5.4	5.9
		PTS	8.9	9.0	7.3	9.8	7.2	7.9	7.5	9.7	SCS	7.4	6.1	6.0	6.8	7.0	5.4	4.7	7.2
	power	11	4.0	3.9	5.4	4.7	5.4	4.0	5.0	4.9	ss	4.5	4.8	4.7	4.4	4.8	4.6	5.1	5.3
		12	5.9	5.1	5.1	5.9	5.3	4.4	4.5	5.0	sc	6.4	5.1	5.1	6.5	4.2	4.9	5.7	5.2
	(P4')	21	29.3	18.4	10.6	43.6	54.7	34.6	20.7	82.1	cs	27.0	16.4	10.3	38.1	46.8	28.8	19.0	69.3
		22	7.4	5.4	5.9	9.9	6.4	5.9	4.0	7.1	cc	5.2	6.1	5.2	6.9	4.1	4.4	3.6	5.2
		PTS	21.2	14.0	10.7	27.7	37.1	20.6	14.0	50.6	SCS	17.5	10.5	9.6	21.4	29.1	15.5	11.3	40.1
		11	5.8	5.3	4.3	4.6	4.3	5.3	4.8	4.8	ss	5.2	4.5	4.7	5.4	5.0	5.8	4.4	4.8
		12	4.7	4.7	5.1	3.8	5.3	5.2	4.9	4.4	sc	4.7	5.3	4.6	4.6	4.7	4.1	4.2	4.1
	(P5')	21	4.6	4.1	6.2	4.2	5.7	5.1	4.9	5.0	cs	5.3	4.3	5.5	5.0	5.8	4.7	4.0	4.7
		22	34.5	4.2	5.5	19.8	54.9	6.2	3.8	41.3	cc	32.2	5.6	4.9	22.7	54.4	7.7	4.4	42.1
		PTS	28.2	5.1	6.2	11.6	46.0	4.2	5.8	16.5	SCS	21.1	5.6	5.5	10.8	37.1	5.1	4.4	16.6
	size	11	5.2	4.4	5.8	5.3	4.6	4.8	4.5	4.5	ss	6.0	4.3	5.2	5.0	4.6	5.7	4.5	4.3
		12	5.3	4.7	4.3	3.9	4.8	6.2	5.0	5.8	sc	5.4	4.9	3.9	5.1	5.2	5.5	5.3	5.6
	(S1')	21	4.8	4.2	4.2	3.9	4.3	5.8	4.6	5.8	cs	5.3	4.5	4.8	5.6	5.2	6.1	7.3	5.7
		22	7.4	7.0	7.5	8.8	6.1	5.5	6.2	8.5	cc	4.9	5.4	6.1	7.2	4.9	4.8	5.6	5.7
		PTS	12.9	8.3	8.8	14.5	8.7	8.7	6.9	10.9	SCS	7.2	5.9	6.1	8.5	4.9	5.9	4.4	6.5
		11	5.3	4.3	5.2	5.0	4.7	5.5	4.9	4.9	ss	5.5	5.8	4.6	5.3	5.4	5.4	4.6	4.4
		12	3.3	4.3	4.7	2.5	5.5	4.7	4.5	3.6	sc	4.6	5.9	5.4	3.8	4.7	4.8	5.5	4.0
	(S2')	21	4.2	4.9	3.3	3.1	3.0	4.3	3.7	4.0	cs	5.2	6.0	5.1	5.6	4.1	4.8	4.4	4.4
		22	6.2	7.4	7.6	9.9	8.3	6.7	7.5	9.3	cc	5.3	5.7	5.9	6.0	6.9	4.3	4.8	6.5
		PTS	8.6	9.2	8.7	9.9	10.0	10.4	8.9	7.6	SCS	5.0	6.5	6.3	6.8	5.9	3.9	4.5	6.4
		11	5.0	5.5	5.7	4.9	5.4	5.2	5.1	4.5	ss	4.6	4.6	4.4	4.8	6.0	5.8	4.3	5.7
		12	5.1	5.5	6.6	4.9	6.2	5.7	6.0	4.4	sc	5.9	4.9	5.9	5.9	5.2	5.4	5.5	4.7
(ii)	(S3')	21	5.0	4.6	5.6	4.9	6.5	5.5	5.4	6.1	cs	4.3	6.0	4.6	4.5	5.5	5.0	4.6	5.3
		22	5.6	6.9	9.3	8.8	7.0	11.3	9.5	11.5	cc	6.6	5.6	6.1	5.3	4.9	6.9	6.4	6.0
		PTS	9.7	7.1	7.1	10.8	8.8	8.9	8.4	12.0	SCS	6.9	6.1	7.0	8.0	5.6	6.4	6.2	6.7
	power	11	5.3	5.5	5.4	6.9	6.0	5.4	4.3	5.5	ss	5.4	4.9	5.7	5.7	5.5	5.1	4.9	5.6
		12	4.6	4.9	4.8	4.3	4.4	5.3	4.5	5.2	sc	5.9	5.0	5.6	5.0	5.2	4.8	3.9	6.4
	(P4')	21	73.6	44.5	24.7	96.1	96.6	74.2	42.3	100.0	cs	68.5	40.3	22.4	91.8	93.5	68.2	38.2	100.0
		22	6.2	5.6	4.9	5.9	5.8	5.7	5.8	7.5	cc	5.4	4.2	4.0	4.9	4.2	6.4	5.2	7.8
		PTS	54.0	30.0	17.2	66.5	89.2	58.7	28.4	99.2	SCS	46.8	24.0	14.1	64.9	81.4	52.0	23.9	97.8
		11	4.6	5.5	4.8	4.1	4.5	5.3	5.1	6.3	ss	5.4	5.9	5.1	4.8	4.4	6.1	4.9	5.3
		12	4.8	3.9	3.6	3.3	5.0	5.6	3.5	5.1	sc	5.0	4.0	4.7	4.9	4.3	5.0	4.4	4.1
	(P5')	21	5.8	4.4	5.5	4.0	5.2	4.0	5.4	4.7	cs	4.9	5.1	5.1	4.7	6.5	4.6	5.8	5.4
		22	46.8	5.0	4.7	29.5	71.1	7.6	5.1	59.1	cc	42.4	7.4	5.6	31.1	70.2	10.0	5.0	61.7
		PTS	34.7	4.9	4.6	9.0	57.1	5.0	5.6	22.9	SCS	27.8	4.9	5.1	11.4	49.3	6.4	4.3	26.2

Notes: Sets (i) and (ii) represent the parameter sets discussed in Section 2 of this appendix.

Table A.5 : The Gaussian QMLEs of GARCH-type models.

		(I)	(II)	(III)	(IV)	(V)	(VI)
	θ_1	0.048	0.024	0.009	- 0.022	0.063	0.019
		(0.012)	(0.012)	(0.029)	(0.028)	(0.019)	(0.018)
	θ_2	0.012	0.019	- 0.048	- 0.051	0.012	0.019
		(0.016)	(0.016)	(0.038)	(0.037)	(0.016)	(0.016)
	θ_3	0.006	- 0.092	0.035	0.017	0.023	- 0.006
		(0.002)	(0.013)	(0.022)	(0.022)	(0.022)	(0.019)
	θ_4	0.938	0.984	0.045	0.046	0.006	- 0.092
		(0.008)	(0.003)	(0.028)	(0.027)	(0.548)	(0.013)
S&P500	θ_5	0.057	- 0.088	- 0.038	- 0.016	0.937	0.984
		(0.008)	(0.012)	(0.071)	(0.043)	(0.007)	(0.003)
	θ_6	·	0.114	0.006	- 0.091	0.058	- 0.088
		·	(0.017)	(0.002)	(0.013)	(0.008)	(0.012)
	θ_7	·	·	0.938	0.983	·	0.114
		·	·	(0.008)	(0.003)	·	(0.017)
	θ_8	·	·	0.057	- 0.088	·	·
		·	·	(0.008)	(0.012)	·	·
	θ_9	·	·	·	0.111	·	·
		·	·	·	(0.017)	·	·
	L_T	-1.288	-1.273	-1.287	-1.273	-1.287	-1.273
	KS	617.235	430.048	613.108	418.513	610.729	443.037
	θ_1	0.066	0.035	- 0.048	- 0.139	0.076	0.040
		(0.015)	(0.015)	(0.033)	(0.040)	(0.017)	(0.017)
	θ_2	0.097	0.104	- 0.011	- 0.050	0.098	0.105
		(0.016)	(0.016)	(0.034)	(0.037)	(0.016)	(0.016)
	θ_3	0.015	- 0.124	0.113	0.082	0.042	0.015
		(0.004)	(0.014)	(0.029)	(0.026)	(0.027)	(0.024)
	θ_4	0.908	0.987	0.082	0.089	0.015	- 0.125
		(0.012)	(0.003)	(0.030)	(0.029)	(0.519)	(0.015)
NASDAQ	θ_5	0.085	- 0.063	0.037	- 0.089	0.907	0.987
		(0.012)	(0.015)	(0.007)	(0.005)	(0.010)	(0.003)
	θ_6	·	0.165	0.014	- 0.122	0.087	- 0.063
		·	(0.019)	(0.004)	(0.015)	(0.012)	(0.015)
	θ_7	·	·	0.910	0.987	·	0.166
		·	·	(0.012)	(0.003)	·	(0.020)
	θ_8	·	·	0.084	- 0.062	·	·
		·	·	(0.012)	(0.015)	·	·
	θ_9	·	·	·	0.161	·	·
		·	·	·	(0.020)	·	·
	L_T	-1.581	-1.572	-1.578	-1.569	-1.580	-1.572
	KS	449.059	367.857	447.849	338.620	436.620	365.173

Notes: L_T and KS represent the fitted value of the log-likelihood function and the Kiefer-Salmon test statistic for conditional normality. The other entries (in the parentheses) are the Gaussian QMLEs (the estimates of the asymptotic standard deviations of the Gaussian QMLEs); see Bollerslev and Wooldridge (1992) for the asymptotic standardized deviations.

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